Understanding QFT path integrals

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1 Functional integrals

We will begin by presenting the notation of for the path integral, and then go into explaining what it means.

$$\int \mathcal{D}\phi \, F[\phi] \tag{1}$$

Here $F[\phi]$ is simple to explain, it is just is a functional of $\phi = \phi(x)$. The unit $\mathcal{D}\phi$ is a bit more convoluted. It is not the ordinary infinitesimal of a Riemann integral[1], instead at this stage you can just think of it as the expression *"over all possible functions"*.

It might be helpful to compare the path integral to the following toy model of the actual integral.

$$\sum_{\substack{\text{All } \phi(x) \text{ with} \\ \text{fixed endpoints}}} F[\phi] \tag{2}$$

So what we are trying to accomplish is to let $\phi(x)$ assume all possible values it can, keeping its endpoints fixed. And then we should sum up all the contributions of $F[\phi]$ from this infinite set of possible functions[2].

Our first worry might be that summing over an infinite set of functional values might not converge. Either the functional must return values very close to zero for the overwhelming majority of $\phi(x)$, or numbers returned by $F[\phi]$ might be cancelled by other contributions with opposite sign (or opposite phase). This seem to profoundly restrict the kind of functionals we can evaluate in this procedure. Luckily, we shall see that the kind of functional we want to consider will not cause the integral to blow up in any way we can not control.

The second worry is to find all those functions. If we intend to sum over them, we have to find some way of enumerating them so we can traverse through all possibilities in a systematic way, and in the end we want to be sure that we have not accidentally excluded any function.

When considering this problem we might realize that this problem would be simpler if we were treating these functions in a computer, where the number of x values representing the function would be a finite set (see figure 1). Also, for a particular x_i, the number of possible functions values $\phi(x_i)$ would also be a finite set – restricted by the resolution and limiting values of the floating point variable. Thus in the end, we can just traverse through every possibility.

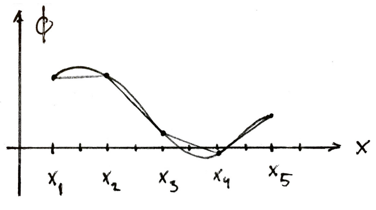

Figure 1: Discrete representation of a continuous function.

[1]Of course, aspiring to mathematical rigour, the Riemann integral is defined as a limit, and not using any kind of infinitesimal. But this is physics so we let this one slide.

[2]Remember, a functional $F[\phi]$ is just a (complex) number.

We can use this idea and take each type of discrete sum to the continuous limit, i.e. integrals. But one caveat arises when we let the number of sample points ($\{x_1, x_2, x_3, \dots\}$, in figure 1) go to infinity. If you work it out you will get a product of an uncountable infinity of integrals in the limit. This seems a little messy, and there is a way to stuff this uncountable infinity in a different part of the calculation, and only deal with a countable infinity of integrals. Clear the floor for function decomposition!

We have fixed endpoints – let us call then x' and x'' – so it will not be too hard to find a complete set of orthogonal functions on the interval. We could for instance use the Fourier series, but we are of course not restricted to that basis as long as the functions are orthogonal and complete. Let $u_i(x)$ be this set of functions that are orthogonal, equation (3a), and complete, equation (3b). The factor N_Ω in equation (3a) is just a normalization factor that appears here since we might not want to use normalized basis functions, for physical reasons.

$$\int_\Omega \mathrm{d}^4x \; u_i(x)\, u_j^*(x) = N_\Omega\, \delta_{ij} \tag{3a}$$

$$\phi(x) = \sum_{i=1}^{\infty} \alpha_i u_i(x) \tag{3b}$$

Thus the uncountable infinity is hidden away in our basis functions $u_i(x)$, and we can specify any function $\phi(x)$ by a *countably* infinite set of Fourier coefficients, α_i.

$$\alpha_i = \int_{x'}^{x''} \mathrm{d}^4x \; u_i^*(x)\, \phi(x) \tag{4}$$

We now have the pieces we require to work towards a definition of the path integral (based on function decomposition). If we decompose $\phi(x)$ according to equation (3b), but we limit ourself to the first n coefficients.

$$\phi(x) \approx \sum_{i=1}^{n} \alpha_i u_i(x) \tag{5}$$

We can then insert this into $F[\phi]$ and integrate over the continuum of possible values for all the α_i.[3] We will throw in some $1/\sqrt{2\pi}$ factors to keep the integral form diverging, we will see how they work their magic later.

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_1}{\sqrt{2\pi}} \cdots \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_n}{\sqrt{2\pi}} \; F\Big[\textstyle\sum_{i=1}^{n} \alpha_i\, u_i(x)\Big] \tag{6}$$

The usefulness of this function decomposition scheme is that we can enumerate the integrals in equation (6), which would not be possible using the method we first discussed (a discrete representation of the function).

As a side note, we can introduce a compact notation for equation (6). Given a certain set of basis functions $\{u_1(x), \cdots, u_n(x)\}$ the functional $F[\phi]$ can

[3]Note that generally α_i are complex numbers, so we might have to integrate over the whole complex plane. Let us still use the notation where the limits are $-\infty$, and ∞, but keep in mind that we might have to extend beyond real numbers. Read more in this Wikipedia article: `https://en.wikipedia.org/wiki/Common_integrals_in_quantum_field_theory`

be replaced by a ordinary function F_{u_n} that takes the Fourier coefficients as arguments.

$$F_{u_n}\big(\alpha_1, \cdots, \alpha_n\big) := F\Big[\textstyle\sum_{i=1}^{n} \alpha_i\, u_i(x)\Big] \quad \Rightarrow \tag{7}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_1}{\sqrt{2\pi}} \cdots \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_n}{\sqrt{2\pi}}\, F\Big[\textstyle\sum_{i=1}^{n} \alpha_i\, u_i(x)\Big] \equiv \prod_{i=0}^{n} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_i}{\sqrt{2\pi}}\, F_{u_n}\big(\alpha_1, \cdots, \alpha_n\big) \tag{8}$$

Finally, we can define the path integral as the limit when we include infinitely many α_i coefficients. Let us spell it out in both notations.

$$\boxed{\int \mathcal{D}\phi\, F[\phi] := \lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_1}{\sqrt{2\pi}} \cdots \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_n}{\sqrt{2\pi}}\, F\Big[\textstyle\sum_{i=1}^{n} \alpha_i\, u_i(x)\Big] \right)} \tag{9}$$

$$\boxed{\int \mathcal{D}\phi\, F[\phi] := \lim_{n\to\infty} \left(\prod_{i=0}^{n} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_i}{\sqrt{2\pi}}\, F_{u_n}\big(\alpha_1, \cdots, \alpha_n\big) \right)} \tag{10}$$

2 Path integrals

In QFT, it turns out, there is a particular functional integral we wish to solve. This is what we call the *path integral.*

$$I_{\mathrm{QFT}} = \int \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}S[\phi]} \tag{11}$$

Here $S[\phi]$ is of curse the familiar action, obtained from the Lagrangian.

$$S[\phi] = \int_{\Omega} \mathrm{d}^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{12}$$

Why is $F[\phi] = \mathrm{e}^{S[\phi]}$ an interesting functional? Let us return to this question when we understand this integral a bit better. We will instead start by looking at a concrete example.

2.1 Path integral over a Klein Gordon field

So we will begin by looking at the path integral in a simple example, the real Klein Gordon field.

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\Big(\partial_\mu \phi\, \partial^\mu \phi - \mu^2 \phi^2\Big) \tag{13}$$

This gives us a specific expression for the action.

$$S[\phi] = \frac{1}{2} \int_{\Omega} \mathrm{d}^4x\, \Big(\partial_\mu \phi\, \partial^\mu \phi - \mu^2 \phi^2\Big) \tag{14}$$

With the purpose of making sense of the path integral of this scalar field action – equation (11) – we will rewrite the action a bit, using integration by parts. Observe that we will require the boundary term to be zero.

$$S[\phi] = \frac{1}{2} \int_{\Omega} \mathrm{d}^4x\, \Big(\partial_\mu \phi\, \partial^\mu \phi - \mu^2 \phi^2\Big) = \tag{15}$$

$$= \frac{1}{2}\left[\phi\,\partial^\mu\phi\right]_\Omega - \frac{1}{2}\int_\Omega \mathrm{d}^4x\,\Big(\phi\,\partial_\mu\partial^\mu\phi - \mu^2\phi^2\Big) = \tag{16}$$

$$= -\frac{1}{2}\int_\Omega \mathrm{d}^4x\,\phi(x)\,\Big(\Box - \mu^2\Big)\phi(x) \tag{17}$$

The next step is to seemingly make this expression somewhat complicated by tacking on a delta function that we integrate over.

$$S[\phi] = -\frac{1}{2}\int_\Omega \mathrm{d}^4x\,\int_\Omega \mathrm{d}^4y\;\phi(x)\,\Big(\Box - \mu^2\Big)_x\,\delta(x-y)\,\phi(y) \tag{18}$$

And lastly we define a propagator-like function which we will use in this final expression for the action.

$$\boxed{K(x-y) := \mathrm{i}\Big(\Box - \mu^2\Big)_x\,\delta(x-y)} \tag{19}$$

$$\boxed{\mathrm{i}S[\phi] = \frac{1}{2}\int_\Omega \mathrm{d}^4x\,\int_\Omega \mathrm{d}^4y\;\phi(x)\,K(x-y)\,\phi(y)} \tag{20}$$

Then we are ready to put together an explicit expression for the path integral in equation (11). Let us use the subscript "KG" for Klein Gordon.

$$I_{\mathrm{KG}} = \int \mathcal{D}\phi\,\mathrm{e}^{\frac{1}{2}\int \mathrm{d}^4x\,\int \mathrm{d}^4y\;\phi(x)\,K(x-y)\,\phi(y)} \tag{21}$$

That was a lot of work for which the purpose is not yet apparent. However, if we look closely at this expression we can see some resemblance to a Gaussian integral. If it were not for the operator $K(x-y)$ that is jammed in between the fields, preventing them from forming a nice square. But suppose that $\phi(x)$ was an eigenfunction to $K(x-y)$. Then the operator would just give us a scalar when acting on the function, and the fields could form a nice square. Of course we could not impose such a tight restriction on $\phi(x)$, however, we can decompose the arbitrary function $\phi(x)$ in the eigenfunctions to the operator K. Luckily the eigenfunctions are just the Fourier basis.

$$\Big(\partial^\mu\partial_\mu - \mu^2\Big)\mathrm{e}^{\mathrm{i}kx} = \cdots = -(k^2 + \mu^2)\,\mathrm{e}^{\mathrm{i}kx} \tag{22}$$

Thus we rewrite the field in terms of its Fourier series.

$$\phi(x) = \frac{1}{\sqrt{V}}\sum_i \alpha_i\,\mathrm{e}^{-\mathrm{i}k_i x} \tag{23}$$

And insert this into equation (21).

$$I_{\mathrm{KG}} = \int \mathcal{D}\phi\,\mathrm{e}^{\frac{1}{2}\int \mathrm{d}^4x\,\int \mathrm{d}^4y\,\left(1/\sqrt{V}\sum_i \alpha_i \mathrm{e}^{-\mathrm{i}k_i x}\right)K(x-y)\left(1/\sqrt{V}\sum_j \alpha_j \mathrm{e}^{-\mathrm{i}k_j y}\right)} = \tag{24}$$

$$= \int \mathcal{D}\phi\,\mathrm{e}^{\frac{1}{2}\sum_{i,j} 1/V\int \mathrm{d}^4x\,\int \mathrm{d}^4y\,\left(\alpha_i \mathrm{e}^{-\mathrm{i}k_i x}\right)K(x-y)\left(\alpha_j \mathrm{e}^{-\mathrm{i}k_j y}\right)} \tag{25}$$

We then wish to use the orthogonality of the basis functions, like in equation (3a). But we have a sign complications in the exponents $e^{-\mathrm{i}k_i x}$, they should be opposite. But remember, we started with real functions $\phi(x)$.

$$\phi(x) = \phi^*(x) \quad \Rightarrow \quad \sum_i \alpha_i \,\mathrm{e}^{-\mathrm{i}k_i x} = \sum_i \alpha_i^* \,\mathrm{e}^{\mathrm{i}k_i x} \tag{26}$$

So we can update equation (25) accordingly.

$$I_{\mathrm{KG}} = \int \mathcal{D}\phi \,\mathrm{e}^{\frac{1}{2}\sum_{i,j} 1/V \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\alpha_i \mathrm{e}^{-\mathrm{i}k_i x}\right) K(x-y) \left(\alpha_j^* \mathrm{e}^{\mathrm{i}k_j y}\right)} = \tag{27}$$

$$= \int \mathcal{D}\phi \,\mathrm{e}^{\frac{1}{2}\sum_{i,j} 1/V \alpha_i \alpha_j^* \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\mathrm{e}^{-\mathrm{i}k_i x} K(x-y)\, \mathrm{e}^{\mathrm{i}k_j y}\right)} \tag{28}$$

We then act with the operator-part of $K(x-y)$ on the eigenfunction just like in equation (22), and then carry out the integrations, first over y and then over x.

$$I_{\mathrm{KG}} = \int \mathcal{D}\phi \,\mathrm{e}^{\frac{1}{2}\sum_{i,j} 1/V \alpha_i \alpha_j^* \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\mathrm{e}^{-\mathrm{i}k_i x}\, \mathrm{i}\delta(x-y)\,(-k^2-\mu^2)\, \mathrm{e}^{\mathrm{i}k_j y}\right)} = \tag{29}$$

$$= \int \mathcal{D}\phi \,\mathrm{e}^{\frac{1}{2}\sum_{i,j} \mathrm{i}/V \alpha_i \alpha_j^* \int \mathrm{d}^4x \left((-k_j^2-\mu^2)\, \mathrm{e}^{\mathrm{i}(k_j-k_i)x}\right)} = \tag{30}$$

$$= \int \mathcal{D}\phi \,\mathrm{e}^{\frac{1}{2}\sum_{i,j} \alpha_i \alpha_j^* -(k_j^2+\mu^2)\, \delta_{k_i k_j}} \tag{31}$$

The Kronecker delta $\delta_{k_i k_j}$ sets $i = j$.

$$I_{\mathrm{KG}} = \int \mathcal{D}\phi \,\mathrm{e}^{-\frac{1}{2}\sum_i (k_i^2+\mu^2)\, |\alpha_i|^2} \tag{32}$$

Now we are ready to use the definition of $\int \mathcal{D}\phi\, F[\phi]$ from equation (9).

$$I_{\mathrm{KG}} = \lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_1}{\sqrt{2\pi}} \cdots \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_n}{\sqrt{2\pi}} \;\mathrm{e}^{-\frac{1}{2}\sum_i (k_i^2+\mu^2)\, |\alpha_i|^2} \right) = \tag{33}$$

$$= \lim_{n\to\infty} \left(\prod_{i=0}^{n} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha_i}{\sqrt{2\pi}} \;\mathrm{e}^{-\frac{1}{2}(k_i^2+\mu^2)\, |\alpha_i|^2} \right) \tag{34}$$

This is an infinite product of Gaussian integrals, each with the familiar solution.

$$\int_{-\infty}^{\infty} \mathrm{d}x \;\mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}} \quad \Rightarrow \tag{35}$$

$$I_{\mathrm{KG}} = \lim_{n\to\infty} \prod_{i=0}^{n} \frac{1}{\sqrt{k_i^2+\mu^2}} = \lim_{n\to\infty} \sqrt{\prod_{i=0}^{n} \frac{1}{k_i^2+\mu^2}} \tag{36}$$

In general, the product of an operators eigenvalues is just the determinant of that operator. Thus we can rewrite equation (36) to a very simple expression.

$$I_{\mathrm{KG}} = \frac{1}{\sqrt{\det K}} \tag{37}$$

It turns out that this result is true in general situations too, and not just for the Klein Gordon field.