Mathematics Companion to Kvantmekanik 1

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Preface

Probably, to many of you, this course is the first serious course in physics. Besides all its conceptual subtleties, that even the most experienced physicists are still struggling with, it also deals with a considerable amount of higher level mathematics. In this companion, I try to gather in one place the mathematics that one needs as a tool to be able to do the calculations that she or he would encounter as we progress in the course. Although, I always enjoy to be mathematically rigorous, but in these notes I don't try and, of course, I don't have the knowledge to be abstract in a mathematical sense of word.

Chapter one is a brief review accompanied by some simple examples of complex numbers. Among these are, introducing polar and exponential representations of a complex number, roots of complex numbers and how to calculate them, and finally three most well-known complex functions, i.e., exponential, sine, and cosine functions.

Chapter two covers the important outlines of integral calculus, specially the method of integration by parts and Gaussian integrals.

Chapter three can be considered as a crash course on Fourier analysis. I have introduced the trigonometric and complex form of the Fourier series of a periodic function. I have also tried to present the Fourier transform as a natural generalization of the Fourier series to the case of non-periodic functions.

In chapter four, I have introduced Dirac delta and step functions in a very informal way, which of course I think it would be enough for this course. These functions are ubiquitous in quantum mechanics.

Finally in the last chapter, important classes of ordinary differential equations of first and second order that are needed in this course and some technical points about them are investigated. Partial differential equations are also introduced and I have tried to highlight the general aspects of the method of separation of variables, which is the only method that we are going to need, by solving a classical textbook example in full detail.

I would also like to mention that, in my opinion the Griffiths' book on quantum mechanics is one of the best books on the subject. Try to read the text and solve its problems as much as you can. It provides you the opportunity of exploring some aspects of quantum mechanics on your own. If you are going to choose physics as your future career, it would be hard to overestimate the importance of quantum mechanics. Trust me, the time you put on it will worth.

Good Luck!

Contents

Chapter 1

Complex Algebra

In this section I recall some important properties of complex numbers, their polar and exponential representations, some points about how to find roots of a complex number, and finally I will include a little about the complex exponential, sine and cosine functions.

1.1 Complex Numbers

I start by defining a complex number and investigate some of the algebraic properties of this system of numbers.

Definition 1.1. A *complex number* z is a number that can be written as $z = x + \mathrm{i}y$ where x and y are real numbers and i is a symbol with the property $\mathrm{i}^2 = -1$.

x is called the *real* part of z and it is denoted by $\mathrm{Re}(z)$ and y is called the *imaginary* part of z and it is denoted by $\mathrm{Im}(z)$.

If $\mathrm{Re}(z) = 0$, z is said to be a *pure imaginary* complex number.

Example 1.1. $2+3\mathrm{i}$, $\frac{1}{2} - \sqrt{2}\mathrm{i}$, and $-5\mathrm{i}$ are examples of complex numbers. The last one is a pure imaginary number.

Note 1.1. *Since any complex number is associated with two real numbers, a complex number $z = x + \mathrm{i}y$ can be represented by the point (x, y) in the xOy plane or a vector whose starting point is the origin and its end point is the point corresponding to (x, y). This is shown in Figure 1. In this case x axis is called the real axis and the y axis is called the imaginary one. In this context, the complex number z and the point z are used interchangeably.*

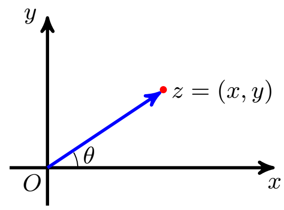

Figure 1.1: Complex Plane

Definition 1.2. A complex number whose real and imaginary parts are both zero, is called the *additive identity* and it is denoted simply by 0.

Definition 1.3. Two complex numbers are said to be *equal* if their real and imaginary parts are correspondingly equal.

Definition 1.4. Let $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ be two complex numbers. Their *sum*, $z_1 + z_2$, their *difference*, $z_1 - z_2$, and their *product*, $z_1 z_2$, are defined as follows,

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + \mathrm{i}(y_1 + y_2) \\ z_1 - z_2 &= (x_1 - x_2) + \mathrm{i}(y_1 - y_2) \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1). \end{aligned} \tag{1.1}$$

Theorem 1.1. *Let z_1, z_2, and z_3 be complex numbers. Then,*

1. $z_1 + z_2 = z_2 + z_1$,
2. $z_1 z_2 = z_2 z_1$,
3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$,
4. $z_1(z_2 z_3) = (z_1 z_2) z_3$,
5. $z_1(z_2 \pm z_3) = z_1 z_2 \pm z_1 z_3$.

Definition 1.5. Let z be a non-zero complex number. Its *reciprocal* is denoted by $\frac{1}{z}$ and it is defined to be the complex number such that $z \cdot \frac{1}{z} = 1$.

Theorem 1.2. *Let $z = x + \mathrm{i}y$ be a non-zero complex number. Then*

$$\frac{1}{z} = \frac{x}{x^2 + y^2} + \mathrm{i}\frac{-y}{x^2 + y^2}. \tag{1.2}$$

Definition 1.6. Let z_1 and z_2 be two complex numbers and $z_2 \neq 0$. The *division* of z_1 by z_2 is denoted by $\frac{z_1}{z_2}$ and it is defined as follows

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}. \tag{1.3}$$

Example 1.2. Simplify the following expression and re-write it in the form of $x + \mathrm{i}y$.

$$\frac{5+5\mathrm{i}}{3-4\mathrm{i}} + \frac{20}{4+3\mathrm{i}}$$

Solution. Invoking the Definition 1.6 and the Theorem 1.2, one can write

$$\begin{aligned}
&\frac{5+5\mathrm{i}}{3-4\mathrm{i}} + \frac{20}{4+3\mathrm{i}} = (5+5\mathrm{i}) \cdot \frac{1}{3-4\mathrm{i}} + 20 \cdot \frac{1}{4+3\mathrm{i}} \\
&\quad = (5+5\mathrm{i}) \cdot \left(\frac{3}{3^2+(-4)^2} + \mathrm{i}\frac{-(-4)}{3^2+(-4)^2} \right) + 20 \cdot \left(\frac{4}{4^2+3^2} + \mathrm{i}\frac{-3}{4^2+3^2} \right) \\
&\quad = 5(1+\mathrm{i}) \cdot \frac{1}{25}(3+4\mathrm{i}) + 20 \cdot \frac{1}{25}(4-3\mathrm{i}) = \frac{1}{5}(3+4\mathrm{i}+3\mathrm{i}+4\mathrm{i}^2) + \frac{4}{5}(4-3\mathrm{i}) \\
&\quad = \frac{1}{5}(-1+7\mathrm{i}+16-12\mathrm{i}) = 3-\mathrm{i}.
\end{aligned}$$

◇

Definition 1.7. Let $z = x + \mathrm{i}y$ be a complex number. Its *conjugate* is denoted by z^* and it is defined as $z^* = x - \mathrm{i}y$.

Theorem 1.3. *Let z be a complex number. Then,*

1. $(z^*)^* = z$,
2. $z + z^* = 2\,\mathrm{Re}(z)$,
3. $z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$.

Theorem 1.4. *Let z_1 and z_2 be two complex numbers. Then,*

1. $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$,
2. $(z_1 z_2)^* = z_1^* z_2^*$,
3. $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$.

Definition 1.8. Let $z = x + \mathrm{i}y$ be a complex number. Its *modulus* is denoted by $|z|$ and it is defined by $|z| = \sqrt{x^2 + y^2}$.

Theorem 1.5. *Let z be a complex number. Then,*

1. $|z^*| = |z|$,
2. $zz^* = |z|^2$,
3. $|z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.

Theorem 1.6. *Let z_1 and z_2 be two complex numbers. Then*

1. $|z_1 z_2| = |z_1||z_2|$,
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $(z_2 \neq 0)$
3. $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$.

Definition 1.9. Let $z = x + \mathrm{i}y$ be a non-zero complex number. Its *principal argument* is denoted by $\mathrm{Arg}(z)$ and it is defined to be the unique properly chosen[§] angle θ in the interval $(-\pi, \pi]$ which the vector associated with z makes with the positive part of the real axis.

Example 1.3. Determine the principal argument of the complex numbers $z_1 = 1 + i$ and $z_2 = \frac{-2}{1-\mathrm{i}\sqrt{3}}$.

Solution. Drawing z_1 in the complex plane, it is easily seen that $\mathrm{Arg}(z_1) = \frac{\pi}{4}$. For z_2, one first needs to re-write z_2 as $x + \mathrm{i}y$. This would be easier to do if one multiplies both the numerator and the denominator of z_2 by the complex conjugate of the denominator as follows

$$\begin{aligned} z_2 &= \frac{-2}{1 - \mathrm{i}\sqrt{3}} \times \frac{1 + \mathrm{i}\sqrt{3}}{1 + \mathrm{i}\sqrt{3}} \\ &= \frac{-2(1 + \mathrm{i}\sqrt{3})}{1^2 + \sqrt{3}^2} \\ &= -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}. \end{aligned}$$

[§]consistent with the region of the complex plane in which the point lies.

Since both real and imaginary parts of z_2 are negative, it resides in the third quadrant of the complex plane and a rough drawing of the vector corresponding to z_2 shows that it makes an angle of $\frac{\pi}{3}$ radians with the negative part of the real axis. Therefore, from Definition 1.9,

$$\begin{aligned}\operatorname{Arg} z_2 &= -\pi + \frac{\pi}{3} \\ &= -\frac{2\pi}{3}.\end{aligned}$$

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1.2 Polar Representation of Complex Numbers

In this section I introduce another representation for a complex number. Consider a non-zero complex number $z = x + \mathrm{i}y$ and let r be its modulus and θ be its principal argument. From Figure 1.1, it is easy to see that $x = r\cos\theta$ and $y = r\sin\theta$. Therefore,

$$z = r(\cos\theta + \mathrm{i}\sin\theta). \tag{1.4}$$

Equation (1.4) is called the *polar representation* of z.

Example 1.4. Consider the complex number $z = \mathrm{i}(1 - \mathrm{i}\sqrt{3})(\sqrt{3} + \mathrm{i})$ and determine its polar representation.

Solution. It would be easier if one first expands z and simplify it. Using $\mathrm{i}^2 = -1$, one can write

$$\begin{aligned}z &= \mathrm{i}(\sqrt{3} + \mathrm{i} - 3\mathrm{i} + \sqrt{3}) \\ &= \mathrm{i}(2\sqrt{3} - 2\mathrm{i}) \\ &= 2 + 2\sqrt{3}\mathrm{i},\end{aligned}$$

so $|z| = 4$ and $\operatorname{Arg}(z) = \frac{\pi}{3}$. Thus, the polar representation of z is

$$z = 4\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right).$$

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Theorem 1.7. (De Moivre's Theorem) *For any integer number n,*

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta. \tag{1.5}$$

Example 1.5. Using De Moivre's theorem, simplify $(1+\mathrm{i})^{100}$.

Solution. Since $|1+\mathrm{i}|=\sqrt{2}$ and $\operatorname{Arg}(1+\mathrm{i})=\frac{\pi}{4}$, the polar representation of $1+\mathrm{i}$ is

$$1+\mathrm{i}=\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right),$$

and therefore,

$$\begin{aligned}(1+\mathrm{i})^{100} &= \left[\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\right]^{100}\\ &= 2^{50}\left(\cos\frac{100\pi}{4}+\mathrm{i}\sin\frac{100\pi}{4}\right)\\ &= 2^{50}(\underbrace{\cos(25\pi)}_{-1}+\mathrm{i}\underbrace{\sin(25\pi)}_{0})\\ &= -2^{50}.\end{aligned}$$

De Moivre's theorem is invoked in the second equality above. ◇

Example 1.6. Using De Moivre's theorem and Newton's binomial expansion, write $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$, respectively.

Solution. From the following Newton's binomial expansion

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$

one gets

$$\begin{aligned}(\cos\theta+\mathrm{i}\sin\theta)^3 &= \cos^3\theta+3(\cos^2\theta)(\mathrm{i}\sin\theta)+3(\cos\theta)(\mathrm{i}\sin\theta)^2+(\mathrm{i}\sin\theta)^3\\ &= (\cos^3\theta-3\cos\theta\sin^2\theta)+\mathrm{i}(3\cos^2\theta\sin\theta-\sin^3\theta),\end{aligned}\tag{1.6}$$

where $\mathrm{i}^2=-1$ and $\mathrm{i}^3=-\mathrm{i}$ have been employed.

On the other hand, by the De Moivre's theorem,

$$(\cos\theta+\mathrm{i}\sin\theta)^3=\cos 3\theta+\mathrm{i}\sin 3\theta.\tag{1.7}$$

Hence by comparing Equations (1.6) and (1.7), one concludes that

$$\cos 3\theta+\mathrm{i}\sin 3\theta=(\cos^3\theta-3\cos\theta\sin^2\theta)+\mathrm{i}(3\cos^2\theta\sin\theta-\sin^3\theta),$$

which, using Definition 1.3, gives rise to the following identities for $\cos 3\theta$ and $\sin 3\theta$,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Of course, using $\sin^2\theta + \cos^2\theta = 1$, these equations can be re-written in the following forms

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \tag{1.8}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta. \tag{1.9}$$

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1.3 nth Roots of a Complex Number

First, the definition of an nth root of a complex number.

Definition 1.10. Let $n \geq 2$ be an integer and z be a complex number. A complex number w is said to be an nth root of z if $w^n = z$.

Theorem 1.8. *Let z be a complex number of modulus r with principal argument θ and let $n \geq 2$ be an integer. There are only n distinct n-th roots w_k of z which can be determined by the following equation,*

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + \mathrm{i}\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1. \tag{1.10}$$

Definition 1.11. w_0, that is the nth root corresponding to $k = 0$ in Equation (1.10), is said to be the *principal nth root* of z.

Note 1.2. *Let z be a complex number and $n \geq 2$ be an integer. The notation $z^{\frac{1}{n}}$ refers to the set of all distinct nth roots of z but $\sqrt[n]{z}$ denotes the principal value of the nth root of z*[§].

Example 1.7. Determine $\sqrt{1+\sqrt{\mathrm{i}}}$.

Solution. Since $|\mathrm{i}| = 1$ and $\mathrm{Arg}(\mathrm{i}) = \frac{\pi}{2}$, from Equation (1.10) the principal second root of i is

$$\sqrt{\mathrm{i}} = \sqrt{1}\left(\cos\frac{\frac{\pi}{2}+0}{2} + \mathrm{i}\sin\frac{\frac{\pi}{2}+0}{2}\right)$$

[§]Of course this is not a universal convention.

$$
\begin{aligned}
&= \cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\\
&= \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\cdot
\end{aligned}
$$

Thus,

$$1+\sqrt{\mathrm{i}} = \left(1+\frac{1}{\sqrt{2}}\right) + \mathrm{i}\frac{1}{\sqrt{2}},$$

and therefore,

$$
\begin{aligned}
|1+\sqrt{\mathrm{i}}| &= \sqrt{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\\
&= \sqrt{2+\sqrt{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Arg}(1+\sqrt{\mathrm{i}}) &= \tan^{-1}\frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}\\
&= \tan^{-1}\frac{1}{\sqrt{2}+1}\\
&= \tan^{-1}(\sqrt{2}-1)\cdot
\end{aligned}
$$

Hence, the principal second root of $1+\sqrt{\mathrm{i}}$ is

$$
\begin{aligned}
\sqrt{1+\sqrt{\mathrm{i}}} &= \sqrt{\sqrt{2+\sqrt{2}}}\left[\cos\left(\frac{\tan^{-1}(\sqrt{2}-1)}{2}\right) + \mathrm{i}\sin\left(\frac{\tan^{-1}(\sqrt{2}-1)}{2}\right)\right]\\
&= \sqrt[4]{2+\sqrt{2}}\left[\cos\left(\frac{\tan^{-1}(\sqrt{2}-1)}{2}\right) + \mathrm{i}\sin\left(\frac{\tan^{-1}(\sqrt{2}-1)}{2}\right)\right].
\end{aligned}
\tag{1.11}
$$

Of course, if you are interested in trigonometry, you can carry on as follows.

Assume that $\tan^{-1}(\sqrt{2}-1) = \theta$, so $\tan\theta = \sqrt{2}-1$ and since $\sqrt{2}-1$ is positive, $0 < \theta < \frac{\pi}{2}$. Using the trigonometric identity

$$1+\tan^2\theta = \frac{1}{\cos^2\theta},$$

one can readily see that

$$\cos\theta = \frac{1}{\sqrt{4-2\sqrt{2}}},$$

or after a little simple algebra

$$\cos\theta = \frac{1}{2}\sqrt{2+\sqrt{2}}\cdot \tag{1.12}$$

Now using Equation (1.12) in the following identities,

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$
$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2},$$

on gets

$$\cos\frac{\theta}{2} = \sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2+\sqrt{2}}} \tag{1.13}$$
$$\sin\frac{\theta}{2} = \sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2+\sqrt{2}}}. \tag{1.14}$$

Eventually, plugging these back into Equation (1.11), we get

$$\sqrt{1+\sqrt{\mathrm{i}}} = \sqrt[4]{2+\sqrt{2}}\left(\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2+\sqrt{2}}}+\mathrm{i}\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2+\sqrt{2}}}\right).$$

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Theorem 1.9. *Let z be a complex number and $n \geq 2$ be an integer. Then*

1. $\left(z^{\frac{1}{n}}\right)^* = (z^*)^{\frac{1}{n}}$

2. $(\sqrt[n]{z})^* = \sqrt[n]{z^*}$

Note 1.3. *One should note that the second part of Theorem 1.9 is not an especial case of the first part. Actually, the equality in the first part is the equality between two sets of complex numbers, each of which have n distinct elements, whereas the second equality is between to specific complex numbers. $(\sqrt[n]{z})^*$ is the conjugate of the principal nth root of z and $\sqrt[n]{z^*}$ is the principal nth root of z^*. The second part asserts the equality of these two complex numbers.*

1.4 Elementary Complex Functions

In this section the exponential, sine, and cosine functions are introduced.

1.4.1 Complex Exponential Function

Definition 1.12. Let $z = x + \mathrm{i}y$. The complex *exponential* function is denoted by e^z and it is defined by

$$\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y). \tag{1.15}$$

Corollary 1.1. *Let θ be a real number. In Equation (1.15), taking x to be zero and y to be θ, one ends up with the following result,*

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta. \tag{1.16}$$

Therefore, a complex number z whose polar representation is $z = r(\cos\theta + \mathrm{i}\sin\theta)$, can also be written as

$$z = r\,\mathrm{e}^{\mathrm{i}\theta}, \tag{1.17}$$

which is called the exponential representation of the complex number z. In this representation, Equation (1.10) takes the following form,

$$w_k = \sqrt[n]{r}\,\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{n}}, \quad k = 0, 1, 2, \ldots, n-1. \tag{1.18}$$

Note 1.4. *One should note that e^z on the left hand side and e^x on the right hand side of Equation (1.15) have different meanings. Since x is real, e^x is exactly the real exponential function which is, a priori, defined in calculus. Of course, the same is true for $\sin y$ and $\cos y$, since y is also real. But e^z is a new concept that is defined in Definition 1.12 which allows us to manipulate the exponential function when the exponent is a complex number. Apparently, if z is a pure real number, i.e. $y = 0$, e^z and e^x coincide, as it is expected of a consistent theory.*

Example 1.8. By calculating $\mathrm{e}^{2+\mathrm{i}\pi}$, show that in contrast to its real counterpart, the complex exponential function may take on negative values.

Solution. By Definition 1.12,

$$\begin{aligned}\mathrm{e}^{2+\mathrm{i}\pi} &= \mathrm{e}^2(\cos\pi + \mathrm{i}\sin\pi)\\ &= \mathrm{e}^2(-1 + \mathrm{i}\times 0)\\ &= -\mathrm{e}^2,\end{aligned}$$

which is a negative number. ◇

Theorem 1.10. *Let $z = x + \mathrm{i}y$ be a complex number. Then*

1. $|\mathrm{e}^z| = \mathrm{e}^x$,

2. $(\mathrm{e}^z)^* = \mathrm{e}^{z^*}$.

Theorem 1.11. *For any complex numbers z and w,*

$$\mathrm{e}^z \mathrm{e}^w = \mathrm{e}^{z+w}, \tag{1.19}$$

and as a special case,

$$\mathrm{e}^{-z} = \frac{1}{\mathrm{e}^z}. \tag{1.20}$$

Theorem 1.12. *Let z and w be two complex numbers. Then $\mathrm{e}^z = \mathrm{e}^w$ if and only if $z - w = 2n\pi\mathrm{i}$ for some integer n.*

Definition 1.13. In quantum mechanics context, an expression $\mathrm{e}^{\mathrm{i}\theta}$, where θ is real, is called a *phase.*

Note 1.5. *From Theorem 1.10, one can see that the modulus of any phase is equal to 1. But one should note that $|\mathrm{e}^{\mathrm{i}z}|$ where z is a complex number is not necessarily equal to one. Actually, if $z = x + iy$,*

$$\begin{aligned}
|\mathrm{e}^{\mathrm{i}z}| &= |\mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)}| \\
&= |\mathrm{e}^{-y+\mathrm{i}x}| \\
&= \mathrm{e}^{-y} \\
&= \mathrm{e}^{-\operatorname{Im}(z)}.
\end{aligned}$$

1.4.2 Complex Sine and Cosine Functions

Definition 1.14. The complex *sine* and *cosine* functions are denote by $\sin z$ and $\cos z$, respectively, and are defined as follows,

$$\begin{aligned}
\sin z &= \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}, \\
\cos z &= \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}.
\end{aligned} \tag{1.21}$$

Note 1.6. *Other trigonometric complex functions are defined the same as their real counterparts, for example*

$$\tan z = \frac{\sin z}{\cos z} \ , \ \cot z = \frac{\cos z}{\sin z}. \tag{1.22}$$

Chapter 2

Integration

In this section I have gathered together important points about different aspects of integration that we face during the course. These include the integral of odd and even functions, differentiation of a definite integral, the method of integration by parts, and a brief review of Gaussian integrals.

2.1 Fundamentals

I start with the fundamental theorem of calculus for definite integrals[§].

Theorem 2.1. *Let f be a continuous function on the interval $[a, b]$ and let F be an* anti-derivative[¶] *of f. Then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{2.1}$$

Note 2.1. *If $a = -\infty$ then $F(a)$ should be interpreted as $\lim_{x\to-\infty} F(x)$ and if $b = +\infty$ then $F(b)$ should be interpreted as $\lim_{x\to+\infty} F(x)$.*

Example 2.1. Determine the value of $\int_{-1}^{3} x^3\,\mathrm{d}x$.

Solution. First we need to determine an anti-derivative of the function $f(x) = x^3$. That is, we need to determine the indefinite integral $\int x^3\,\mathrm{d}x$ which, using the

[§]Integrals with lower and upper limits.

[¶]F is said to be an anti-derivative of f on an interval I, if for any x in I, $F'(x) = f(x)$.

following formula[§],

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1}, \quad (n \neq -1),$$

is equal to $F(x) := \frac{x^4}{4}$. Thus, from Theorem 2.1,

$$\begin{aligned}\int_{-1}^{3} x^3 \,\mathrm{d}x &= F(3) - F(-1) \\ &= \frac{3^4}{4} - \frac{(-1)^4}{4} \\ &= 20.\end{aligned}$$

$\diamond$

Note 2.2. *The integration variable in a definite integral is a dummy variable, i.e., changing it to another variable does not affect the result. For instance, in Example 2.1 the value of* $\int_{-1}^{3} t^3 \,\mathrm{d}t$ *is also 20.*

Example 2.2. Let a be a positive real number. Determine $\int_0^{+\infty} \mathrm{e}^{-ax} \,\mathrm{d}x$.

Solution. Using the formula,

$$\int \mathrm{e}^{\alpha x} \,\mathrm{d}x = \frac{1}{\alpha} \,\mathrm{e}^{\alpha x},$$

Theorem 2.1 and Note 2.1, one gets

$$\begin{aligned}\int_0^{+\infty} \mathrm{e}^{-ax} \,\mathrm{d}x &= \left[\lim_{x \to +\infty} \left(\frac{1}{-a} \,\mathrm{e}^{-ax}\right)\right] - \frac{1}{-a} \,\mathrm{e}^{0x} \\ &= 0 + \frac{1}{a} \\ &= \frac{1}{a}.\end{aligned}$$

Note that, since a is positive, $\lim_{x \to +\infty} \mathrm{e}^{-ax} = 0$. $\diamond$

Example 2.3. Let a be a positive real number. Compute $\int_0^{+\infty} r\mathrm{e}^{-ar^2} \,\mathrm{d}r$.

Solution. First, I calculate the indefinite integral $\int r\mathrm{e}^{-ar^2} \,\mathrm{d}r$ by the following substitution to a new variable R,

$$r^2 = R.$$

[§]Suppressing the constant of integration which, when calculating the definite integrals, does not affect the result.

Differentiating both sides, yields

$$2r\,\mathrm{d}r = \mathrm{d}R,$$

or

$$r\,\mathrm{d}r = \frac{1}{2}\,\mathrm{d}R.$$

On the other hand, since the integration interval of r is $(0, +\infty)$, and $R = r^2$, the integration interval of R is also $(0, +\infty)$ and therefore,

$$\begin{aligned}\int_0^{+\infty} r\mathrm{e}^{-ar^2}\,\mathrm{d}r &= \int_0^{+\infty} \mathrm{e}^{-ar^2}\underbrace{r\,\mathrm{d}r}_{\frac{1}{2}\,\mathrm{d}R}\\ &= \frac{1}{2}\int_0^{+\infty} \mathrm{e}^{-aR}\,\mathrm{d}R\\ &= \frac{1}{2a}.\end{aligned}$$

In the last step, I used Note 2.2 and the result of Example 2.2. ◇

Definition 2.1. If $a > b$, then

$$\int_a^b f(x)\,\mathrm{d}x := -\int_b^a f(x)\,\mathrm{d}x. \tag{2.2}$$

Corollary 2.1. *An immediate consequence of Definition 2.1 is that*

$$\int_a^a f(x)\,\mathrm{d}x = 0. \tag{2.3}$$

Theorem 2.2. (Linearity of the Definite Integrals) *Let f and g be two continuous functions on the interval $[a, b]$ and let α and β be two constants*[§]*. Then*

$$\int_a^b [\alpha f(x) + \beta g(x)]\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x + \beta\int_a^b g(x)\,\mathrm{d}x. \tag{2.4}$$

Theorem 2.3. (Decomposition of the Definite Integral) *Let f be a continuous function on the interval $[a, b]$ and let c be between a and b. Then*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \tag{2.5}$$

[§] i.e., independent of x.

Note 2.3. *Theorems 2.2 and 2.3 are also valid if* $a = -\infty$ *and/or* $b = +\infty$.

Note 2.4. *By Definition 2.1, one can easily show that Theorem 2.3 is also valid even if* c *is not between* a *and* b.

Theorem 2.4. *Let* f *be a continuous function on the whole real line. Then,*

$$\int_a^b f(x)\,\mathrm{d}x = \int_{a\pm c}^{b\pm c} f(x \mp c)\,\mathrm{d}x, \tag{2.6}$$

for any real numbers a, b, *and* c.

Proof. Let $X = x \mp c$, so $\mathrm{d}X = \mathrm{d}x$ and, since $x \in (a \pm c, b \pm c)$, $X \in (a, b)$. But, since X is a dummy variable, one can change X to any other variable, including x itself. This shows that the right hand side of Equation (2.6) is equal to its left hand side. ■

Definition 2.2. Let f be a function of domain D_f such that for every x in D_f, $-x$ is also in D_f. Then,

1. f is said to be an *odd* function, if for all x in D_f, $f(-x) = -f(x)$.
2. f is said to be an *even* function, if for all x in D_f, $f(-x) = f(x)$.

Example 2.4. Based on the Definition 2.2, it is readily seen that $f(x) = \sin x$ is an odd function, $g(x) = x^3 \sin x$ is an even function, but $h(x) = x^2 + x^3$ is neither an odd function nor an even one.

Theorem 2.5. *Let* f *be an odd and* g *be an even function on the interval* $[-a, a]$. *Then*

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0, \tag{2.7}$$

and

$$\int_{-a}^{a} g(x)\,\mathrm{d}x = 2\int_0^a g(x)\,\mathrm{d}x. \tag{2.8}$$

Note 2.5. *Theorem 2.5 is also valid if* $a = \infty$.

Example 2.5. Compute the following definite integral,

$$\int_{-\infty}^{+\infty} (x^3 + x^5)\mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x.$$

Solution. Let $f(x) = (x^3 + x^5)\mathrm{e}^{-\frac{1}{2}x^2}$. The domain of f is $\mathbb{R}$ and so for every x in the domain, $-x$ is also in the domain. Moreover,

$$\begin{aligned} f(-x) &= \left[(-x)^3 + (-x)^5\right] \mathrm{e}^{-\frac{1}{2}(-x)^2} \\ &= -(x^3 + x^5)\mathrm{e}^{-\frac{1}{2}x^2} \\ &= -f(x)\cdot \end{aligned}$$

Hence, f is an odd function and thus, by Note 2.5 and Theorem 2.5, this integral vanishes. ◇

Example 2.6. Using Theorem 2.1, show that $\int_0^3 (x^2t^3 + 2xt)\,\mathrm{d}x$ is a function of t and determine its derivative at $t = 1$.

Solution. $\mathrm{d}x$ indicates that the integration variable is x, so the other variable t can be treated as a constant. By linearity of the definite integrals, one can write

$$\begin{aligned} \int_0^3 (x^2t^3 + 2xt)\,\mathrm{d}x &= t^3 \int_0^3 x^2\,\mathrm{d}x + 2t \int_0^3 x\,\mathrm{d}x \\ &= t^3 \left(\frac{x^3}{3}\right)\Big|_0^3 + 2t\left(\frac{x^2}{2}\right)\Big|_0^3 \\ &= 9t^3 + 9t. \end{aligned}$$

As is clear from the above calculations, the given integral is a function of t. Thus,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^3 (x^2t^3 + 2xt)\,\mathrm{d}x\Bigg|_{t=1} &= \frac{\mathrm{d}}{\mathrm{d}t}(9t^3 + 9t)\Big|_{t=1} \\ &= (27t^2 + 9)|_{t=1} \\ &= 36. \end{aligned}$$

◇

Note 2.6. *As Example 2.6 motivates, $\int_a^b f(x,t)\,\mathrm{d}x$ is a function of t and Theorem 2.6 provides a formula to determine its derivative without first calculating the integral.*

Theorem 2.6. *Let $f(x,t)$ be a "well-behaved"[§] function. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x. \tag{2.9}$$

Note 2.7. *Theorem 2.6 is also valid if $a = -\infty$ and/or $b = +\infty$.*

Corollary 2.2. *The above theorem can easily be generalized for higher order derivatives as*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \frac{\partial^n f}{\partial t^n}(x,t)\,\mathrm{d}x. \tag{2.10}$$

Example 2.7. Solve the second part of Example 2.6 again, but this time make use of Theorem 2.6.

Solution. By Theorem 2.6,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^3 (x^2t^3 + 2xt)\,\mathrm{d}x\Bigg|_{t=1} &= \int_0^3 \frac{\partial}{\partial t}(x^2t^3 + 2xt)\,\mathrm{d}x\Bigg|_{t=1} \\
&= \int_0^3 (3t^2x^2 + 2x)\,\mathrm{d}x\Bigg|_{t=1} \\
&= \int_0^3 [3(1)^2x^2 + 2x]\,\mathrm{d}x \\
&= 3\int_0^3 x^2\,\mathrm{d}x + 2\int_0^3 x\,\mathrm{d}x \\
&= 3 \times 9 + 2 \times \frac{9}{2} \\
&= 36.
\end{aligned}$$

◇

Theorem 2.7. *Let a be a positive real number and n be a non-negative integer. Then,*

$$\int_0^{+\infty} x^n \mathrm{e}^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}}\cdot \tag{2.11}$$

[§] By "well-behaved" I mean a set of abstract mathematical conditions on $f(x,t)$, like continuity and existence of some integrals and derivatives, ..., which is always assumed to be valid in physical problems. Therefore, I do not bother you by mentioning them explicitly.

Proof. In Example 2.2, we saw that

$$\int_0^{+\infty} \mathrm{e}^{-ax}\,\mathrm{d}x = \frac{1}{a},$$

so

$$\frac{\mathrm{d}^n}{\mathrm{d}a^n}\int_0^{+\infty} \mathrm{e}^{-ax}\,\mathrm{d}x = \frac{\mathrm{d}^n}{\mathrm{d}a^n}\left(\frac{1}{a}\right)\cdot \tag{2.12}$$

By Corollary 2.2,

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}a^n}\int_0^{+\infty} \mathrm{e}^{-ax}\,\mathrm{d}x &= \int_0^{+\infty} \frac{\partial^n}{\partial a^n}\left(\mathrm{e}^{-ax}\right)\,\mathrm{d}x\\ &= \int_0^{+\infty} (-x)^n \mathrm{e}^{-ax}\,\mathrm{d}x\\ &= (-1)^n \int_0^{+\infty} x^n \mathrm{e}^{-ax}\,\mathrm{d}x. \end{aligned}\tag{2.13}$$

On the other hand,

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}a^n}\left(\frac{1}{a}\right) &= \frac{\mathrm{d}^n}{\mathrm{d}a^n}(a^{-1})\\ &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}a^{n-1}}(-1\times a^{-2})\\ &= \frac{\mathrm{d}^{n-2}}{\mathrm{d}a^{n-2}}(-1\times -2\times a^{-3})\\ &= \cdots\\ &= -1\times -2\times -3\times\cdots\times -n\times a^{-1-n}\\ &= (-1)^n\, n!\; a^{-(n+1)}\\ &= (-1)^n\,\frac{n!}{a^{n+1}}\cdot \end{aligned}\tag{2.14}$$

Plugging Equations (2.13) and (2.14) back into Equation (2.12) proves the theorem. ■

Note 2.8. *If $a \leq 0$, then the integral given in Theorem 2.7 diverges.*

Example 2.8. Determine the value of the following integral

$$J = \int_0^{+\infty} x^7 \mathrm{e}^{-2x^2}\,\mathrm{d}x.$$

Solution. J can be re-written as

$$J = \int_0^{+\infty} (x^2)^3 \mathrm{e}^{-2x^2} (x\,\mathrm{d}x),$$

which by introducing the new variable $X = x^2$ can be transformed into

$$\begin{aligned} J &= \int_0^{+\infty} X^3 \mathrm{e}^{-2X} \left(\frac{1}{2}\,\mathrm{d}X\right) \\ &= \frac{1}{2}\int_0^{+\infty} X^3 \mathrm{e}^{-2X}\,\mathrm{d}X \\ &= \frac{1}{2}\cdot\frac{3!}{2^4} \\ &= \frac{3}{16}. \end{aligned}$$

In the last step, I used Theorem 2.7 for $n = 3$ and $a = 2$. ◇

2.2 The Method of Integration by Parts

Integration by parts is one of the powerful methods for calculating definite and indefinite integrals.

Theorem 2.8. *Let f, g, f', and g' be continuous on the interval $[a, b]$. Then,*

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x. \tag{2.15}$$

Note 2.9. *Theorem 2.8 is also valid if $a = -\infty$ and/or $b = +\infty$. Of course, in this case the first term in Equation (2.15), which is called a boundary term, should be determined by calculating the appropriate limits.*

Example 2.9. Determine the value of $J = \int_0^\pi x \sin x\,\mathrm{d}x$.

Solution. Since $(\cos x)' = -\sin x$, J can be written as

$$J = \int_0^\pi x(-\cos x)'\,\mathrm{d}x.$$

So taking $f(x) = x$ and $g(x) = -\cos x$ in Equation (2.15) yields,

$$J = x(-\cos x)\big|_0^\pi - \int_0^\pi (x)'(-\cos x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= [\pi(-\cos\pi) - 0(-\cos 0)] + \int_0^{\pi} \cos x \,\mathrm{d}x \\
&= \pi + \sin x|_0^{\pi} \\
&= \pi.
\end{aligned}
$$

◇

Example 2.10. Compute the value of $J = \int_1^{\mathrm{e}} \ln x \,\mathrm{d}x$.

Solution. J can be written as

$$J = \int_1^{\mathrm{e}} \ln x \,(x)' \,\mathrm{d}x.$$

Thus, plugging $f(x) = \ln x$ and $g(x) = x$ in Equation (2.15) and noting that $(\ln x)' = \frac{1}{x}$, gives

$$
\begin{aligned}
J &= (\ln x)x|_1^{\mathrm{e}} - \int_1^{\mathrm{e}} (\ln x)' \,x \,\mathrm{d}x \\
&= [(\ln \mathrm{e})\mathrm{e} - (\ln 1)1] - \int_1^{\mathrm{e}} \mathrm{d}x \\
&= (e - 0) - (\mathrm{e} - 1) \\
&= 1.
\end{aligned}
$$

◇

Example 2.11. Prove the following identity:

$$\int_a^b f(x)g''(x)\,\mathrm{d}x = f(x)g'(x)|_a^b - f'(x)g(x)|_a^b + \int_a^b f''(x)g(x)\,\mathrm{d}x.$$

Solution. As follows, this can be done easily by applying the Equation (2.15) on the left hand side (L.H.S.) of this identity twice,

$$
\begin{aligned}
\text{L.H.S.} &= \int_a^b f(x)\,(g'(x))'\,\mathrm{d}x \\
&= f(x)g'(x)|_a^b - \int_a^b f'(x)g'(x)\,\mathrm{d}x \\
&= f(x)g'(x)|_a^b - \left(f'(x)g(x)|_a^b - \int_a^b f''(x)g(x)\,\mathrm{d}x \right)
\end{aligned}
$$

$$= f(x)g'(x)|_a^b - f'(x)g(x)|_a^b + \int_a^b f''(x)g(x)\,\mathrm{d}x$$
$$= \text{R.H.S..}$$

◇

Note 2.10. *The identity in Example 2.11 can be generalized in a similar manner to include higher order derivatives as follows:*

$$\int_a^b f(x)g^{(n)}(x)\,\mathrm{d}x = f(x)g^{(n-1)}(x)|_a^b - f^{(1)}(x)g^{(n-2)}(x)|_a^b + f^{(2)}(x)g^{(n-3)}(x)|_a^b - \cdots + (-1)^{n-1}f^{(n-1)}(x)g(x)|_a^b + (-1)^n \int_a^b f^{(n)}(x)g(x)\,\mathrm{d}x, \quad (2.16)$$

where the superscript on a function refers to the order of the derivative of that function. Equation (2.16) can be considered in a more practical way as

$$\int_a^b F \cdot G\,\mathrm{d}x = F \cdot I(G)|_a^b - D(F)\cdot I^2(G)|_a^b + D^2(F)\cdot I^3(G)|_a^b - \cdots + (-1)^{n-1}D^{n-1}(F)\cdot I^n(G)|_a^b + (-1)^n \int_a^b D^n(F)\cdot I^n(G)\,\mathrm{d}x, \quad (2.17)$$

where $D^k(F)$ denotes the kth order derivative of F and $I^k(G)$ refers to the function which is obtained after k times integrating G consecutively. One should note that in this formula n can be any arbitrarily chosen positive integer.

Example 2.12. Use Equation 2.16 to determine the value of $J = \int_0^\pi x^3 \sin x\,\mathrm{d}x$.

Solution. Since $(\cos x)''' = \sin x$, J can be written as

$$J = \int_0^\pi x^3\,(\cos x)'''\,\mathrm{d}x.$$

Therefore, plugging $f(x) = x^3$, $g(x) = \cos x$, and $n = 3$ in Equation (2.16), one gets

$$J = x^3(\cos x)''|_0^\pi - (x^3)'(\cos x)'|_0^\pi + (x^3)''(\cos x)|_0^\pi - \int_0^\pi (x^3)'''\cos x\,\mathrm{d}x$$

$$
\begin{aligned}
&= x^3(-\cos x)\big|_0^\pi - (3x^2)(-\sin x)\big|_0^\pi + (6x)(\cos x)\big|_0^\pi - \int_0^\pi 6\cos x \,\mathrm{d}x \\
&= \pi^3 - 0 + (-6\pi) - 6\sin x\big|_0^\pi \\
&= \pi^3 - 6\pi.
\end{aligned}
$$

Of course, one can also use Equation (2.17) to calculate J by letting $F(x) = x^3$, $G(x) = \sin x$, and choosing n to be 4 as follows

$$
\begin{aligned}
J &= x^3(-\cos x)\big|_0^\pi - (3x^2)(-\sin x)\big|_0^\pi + (6x)(\cos x)\big|_0^\pi - (6)(\sin x)\big|_0^\pi \\
&= \pi^3 - 6\pi.
\end{aligned}
$$

$\diamond$

Note 2.11. *If all the boundary terms in Equation (2.16) are zero, then this equation reduces to the following equation which is useful when dealing with wave functions in quantum mechanics,*

$$
\boxed{\int_a^b f(x)g^{(n)}(x)\,\mathrm{d}x = (-1)^n \int_a^b f^{(n)}(x)g(x)\,\mathrm{d}x.} \tag{2.18}
$$

Theorem 2.9. *Let n be a non-negative integer. Then*

$$
\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta\,\mathrm{d}\theta = \frac{2^{2n}\,(n!\,)^2}{(2n+1)!}. \tag{2.19}
$$

Proof. I prove the first one. The proof for the other one is completely similar. Since it can easily be checked that

$$
\int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta = 1,
$$

it is obvious that Equation (2.19) is valid for $n = 0$. Now suppose that $n \geq 1$ and let

$$
J_n = \int_0^{\pi} \sin^{2n+1}\theta\,\mathrm{d}\theta. \tag{2.20}
$$

So one can write J_n as

$$
J_n = \int_0^{\frac{\pi}{2}} \sin^{2n}\theta(-\cos\theta)'\,\mathrm{d}\theta,
$$

and now using integration by parts,

$$
\begin{aligned}
J_n &= \underbrace{\sin^{2n}\theta(-\cos\theta)\big|_0^{\frac{\pi}{2}}}_{0} - \int_0^{\frac{\pi}{2}} \underbrace{\left(\sin^{2n}\theta\right)'}_{2n\sin^{2n-1}\theta\cos\theta} (-\cos\theta)\,\mathrm{d}\theta \\
&= 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\cos^2\theta\,\mathrm{d}\theta \\
&= 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta(1-\sin^2\theta)\,\mathrm{d}\theta \\
&= 2n\underbrace{\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,\mathrm{d}\theta}_{J_{n-1}} - 2n\underbrace{\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\,\mathrm{d}\theta}_{J_n}.
\end{aligned}
$$

Thus,

$$J_n = 2n\,J_{n-1} - 2n\,J_n,$$

or, for all $n \geq 1$,

$$\frac{J_n}{J_{n-1}} = \frac{2n}{2n+1}\cdot$$

Hence,

$$\frac{J_1}{J_0} = \frac{2}{3}\,,\ \frac{J_2}{J_1} = \frac{4}{5}\,,\ \frac{J_3}{J_2} = \frac{6}{7}\,,\ \cdots\,,\ \frac{J_n}{J_{n-1}} = \frac{2n}{2n+1}\cdot$$

After multiplying the corresponding sides of these fractions and canceling out the extra terms, one gets

$$
\begin{aligned}
\frac{J_n}{J_0} &= \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)} \\
&= \frac{(2\cdot 4\cdot 6\cdots(2n))^2}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots(2n)(2n+1)} \\
&= \frac{\left(2^n\cdot 1\cdot 2\cdot 3\cdots n\right)^2}{(2n+1)!} \\
&= \frac{2^{2n}(n!\,)^2}{(2n+1)!}\cdot
\end{aligned}
$$

Since, $J_0 = \int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta = 1$, the above equation yields the result. ■

Example 2.13. Let n be a non-negative integer. Show that

$$\int_0^\pi \sin^{2n+1}\theta \,\mathrm{d}\theta = \frac{2^{2n+1}(n!)^2}{(2n+1)!}, \tag{2.21}$$

and

$$\int_0^\pi \cos^{2n+1}\theta \,\mathrm{d}\theta = 0. \tag{2.22}$$

Solution. Let I_n be the integral in Equation (2.21), H_n be the integral in Equation (2.22), and J_n be the integral in Equation (2.20). I prove that $I_n = 2J_n$ and $H_n = 0$. By Theorems 2.3 and 2.4,

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \,\mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi} \sin^{2n+1}\theta \,\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \sin^{2n+1}\left(\theta + \frac{\pi}{2}\right) \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \,\mathrm{d}\theta. \end{aligned}$$

But, by Theorem 2.9, the last two integrals above are equal and therefore,

$$I_n = 2J_n.$$

For H_n, using the same Theorems 2.3 and 2.4 again, we have

$$\begin{aligned} H_n &= \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \,\mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi} \cos^{2n+1}\theta \,\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \,\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \cos^{2n+1}\left(\theta + \frac{\pi}{2}\right) \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \,\mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \,\mathrm{d}\theta \\ &= 0. \end{aligned}$$

◇

Example 2.14. Let n be a non-negative integer. Determine the following integral

$$I_n = \int_{-1}^{1} (1 - x^2)^n \,\mathrm{d}x.$$

Solution. Since the integrand is an even function, by Theorem (2.5),

$$I_n = 2K_n, \tag{2.23}$$

where

$$K_n = \int_0^1 (1-x^2)^n \,\mathrm{d}x. \tag{2.24}$$

Now let $x = \sin\theta$, thus $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ and, since x in K_n belongs to the interval $(0,1)$, θ belongs to the interval $\left(0, \frac{\pi}{2}\right)$. Hence,

$$\begin{aligned} K_n &= \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)^2 (\cos\theta\,\mathrm{d}\theta) \\ &= \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta\,\mathrm{d}\theta, \end{aligned}$$

where $\sin^2\theta + \cos^2\theta = 1$ is used. Thus, by Theorem 2.9 and Equation (2.23),

$$I_n = \frac{2^{2n+1}(n!\,)^2}{(2n+1)!}.$$

◇

2.3 Gaussian Integrals

There is a class of integrals, known as Gaussian integrals, which appears a lot in this course. The goal of this section is to calculate this kind of integrals.

Definition 2.3. Any integral of the general form

$$\int_0^{+\infty} x^k \mathrm{e}^{-ax^2}\,\mathrm{d}x,$$

where k is a non-negative integer and a is a positive real number is called a *Gaussian* integral[§].

First let us look at the easiest case where k is equal to zero.

Lemma 2.1. *Let a be a positive real number. Then,*

$$\int_0^{+\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}. \tag{2.25}$$

[§] If k is not an integer, a Gaussian integral can be expressed in terms of some special functions, called *Gamma* functions. But in this course we don't need them.

Proof. Let

$$J = \int_0^{+\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x.$$

Since the integration variable is a dummy variable, one can also write

$$J = \int_0^{+\infty} \mathrm{e}^{-ay^2}\,\mathrm{d}y.$$

Therefore,

$$J^2 = \left(\int_0^{+\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x\right)\left(\int_0^{+\infty} \mathrm{e}^{-ay^2}\,\mathrm{d}y\right).$$

By Fubini's theorem[§], this can be written as

$$\begin{aligned} J^2 &= \int_0^{+\infty}\int_0^{+\infty} \mathrm{e}^{-ax^2}\mathrm{e}^{-ay^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{+\infty}\int_0^{+\infty} \mathrm{e}^{-a(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{2.26}$$

which is a double integral over the first quadrant. Now we introduce the polar coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

Hence, $x^2 + y^2 = r^2$ and from what is known in calculus, $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$[¶]. In the first quadrant r goes from 0 to $+\infty$ and θ goes from 0 to $\frac{\pi}{2}$. Thus, Equation (2.26) can be written as

$$\begin{aligned} J^2 &= \int_0^{+\infty}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-ar^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{+\infty} r\mathrm{e}^{-ar^2}\,\mathrm{d}r \end{aligned}$$

[§] Let $f(x,y) = g(x)h(y)$ be a continuous function on the region $D = [a,b]\times[c,d]$ of the xOy plane. Then

$$\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

[¶] Actually, r appearing here, as the pre-factor, is the Jacobian of the polar transformation.

$$
\begin{aligned}
&= \frac{\pi}{2} \cdot \frac{1}{2a} \\
&= \frac{1}{4} \frac{\pi}{a}.
\end{aligned}
$$

In the third line above, I employed the result of Example 2.3. Now taking the square roots we have

$$J = \pm \frac{1}{2} \sqrt{\frac{\pi}{a}},$$

but, since the integrand e^{-ax^2} of J is an always-positive function, J should be non-negative[§], and

$$J = \frac{1}{2} \sqrt{\frac{\pi}{a}}.$$

■

Corollary 2.3. *Since* e^{-ax^2} *is an even function,*

$$\boxed{\int_{-\infty}^{+\infty} \mathrm{e}^{-ax^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{a}}} \tag{2.27}$$

which we use it a lot during the course.

Note 2.12. *Using the techniques in complex analysis, one can see that Equations (2.25) and (2.27) are also valid if* a *is a pure imaginary complex number or a complex number with a positive real part. In the latter case,* $\sqrt{a}$ *refers to the principal square root of* a*.*

Now let us consider the case where k is a positive even integer, say, $k = 2n$.

Theorem 2.10. *Let* n *be a positive integer and* a *be a positive real number. Then,*

$$\int_{0}^{+\infty} x^{2n} \mathrm{e}^{-ax^2} \, \mathrm{d}x = \frac{(2n)!}{n!} \frac{1}{2^{2n+1}} \frac{1}{a^n} \sqrt{\frac{\pi}{a}}. \tag{2.28}$$

[§]Let f be a function which is continuous on the interval $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Then

$$\int_a^b f(x) \, \mathrm{d}x \geq 0.$$

Proof. Let

$$J = \int_0^{+\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x.$$

By Corollary 2.2,

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}a^n} J &= \int_0^{+\infty} \frac{\partial^n}{\partial a^n}\left(\mathrm{e}^{-ax^2}\right)\,\mathrm{d}x \\
&= \int_0^{+\infty} (-x^2)^n \mathrm{e}^{-ax^2}\,\mathrm{d}x \\
&= (-1)^n \int_0^{+\infty} x^{2n} \mathrm{e}^{-ax^2}\,\mathrm{d}x. \qquad (2.29)
\end{aligned}$$

On the other hand, by Lemma 2.1, $J = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, so

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}a^n} J &= \frac{\sqrt{\pi}}{2}\,\frac{\mathrm{d}^n}{\mathrm{d}a^n}\left(a^{-\frac{1}{2}}\right) \\
&= \frac{\sqrt{\pi}}{2}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}a^{n-1}}\left(-\frac{1}{2}\times a^{-\frac{1}{2}-1}\right) \\
&= \frac{\sqrt{\pi}}{2}\,\frac{\mathrm{d}^{n-2}}{\mathrm{d}a^{n-2}}\left[-\frac{1}{2}\times\left(-\frac{1}{2}-1\right)\times a^{-\frac{1}{2}-2}\right] \\
&= \cdots \\
&= \frac{\sqrt{\pi}}{2}\left[-\frac{1}{2}\times\left(-\frac{1}{2}-1\right)\times\cdots\times\left(-\frac{1}{2}-(n-1)\right)\times a^{-\frac{1}{2}-n}\right] \\
&= \frac{\sqrt{\pi}}{2}(-1)^n\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\right)\frac{1}{a^n\sqrt{a}} \\
&= (-1)^n\,\frac{1}{a^n}\sqrt{\frac{\pi}{a}}\,\frac{1\cdot 3\cdots(2n-1)}{2^{n+1}}. \qquad (2.30)
\end{aligned}$$

The product in the numerator can be written as

$$\begin{aligned}
1\cdot 3\cdots(2n-1) &= \frac{1\cdot 2\cdot 3\cdots(2n-1)(2n)}{2\cdot 4\cdots(2n)} \\
&= \frac{(2n)!}{2^n(1\cdot 2\cdots n)} \\
&= \frac{(2n)!}{2^n\,n!}.
\end{aligned}$$

Replacing this result back into Equation (2.30) gives rise to

$$\frac{\mathrm{d}^n}{\mathrm{d}a^n} J = (-1)^n \frac{(2n)!}{n!} \frac{1}{2^{2n+1}} \frac{1}{a^n} \sqrt{\frac{\pi}{a}}. \tag{2.31}$$

Equating both sides of Equations (2.29) and (2.31), $(-1)^n$ cancels out and the goal is achieved. ■

Corollary 2.4. *Again, since the function $x^{2n}\mathrm{e}^{-ax^2}$ is an even function,*

$$\int_{-\infty}^{+\infty} x^{2n}\mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{(2n)!}{n!} \frac{1}{2^{2n}} \frac{1}{a^n} \sqrt{\frac{\pi}{a}}. \tag{2.32}$$

Note 2.13. *The $n = 1$ case of the Corollary 2.4 which reads as*

$$\boxed{\int_{-\infty}^{+\infty} x^{2}\mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{2a} \sqrt{\frac{\pi}{a}}} \tag{2.33}$$

is widely used in this course.

Finally, let k be an odd positive integer.

Theorem 2.11. *Let n be a positive integer and a be a positive real number. Then,*

$$\int_{0}^{+\infty} x^{2n+1}\mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{n!}{2a^{n+1}}. \tag{2.34}$$

Proof. Let's call the given integral as J_n. J_n can be written as

$$J_n = \int_0^{+\infty} x^{2n}\left(x\mathrm{e}^{-ax^2}\right)\,\mathrm{d}x,$$

and since,

$$x\mathrm{e}^{-ax^2} = -\frac{1}{2a}\left(\mathrm{e}^{-ax^2}\right)',$$

we have,

$$J_n = -\frac{1}{2a}\int_0^{+\infty} x^{2n}\left(\mathrm{e}^{-ax^2}\right)'\,\mathrm{d}x.$$

Using integration-by-part method, it follows

$$J_n = -\frac{1}{2a}\left(x^{2n}\mathrm{e}^{-ax^2}\Big|_0^{+\infty} - \int_0^{+\infty}\left(x^{2n}\right)'\mathrm{e}^{-ax^2}\,\mathrm{d}x\right)$$

$$
\begin{aligned}
&= -\frac{1}{2a}\left(\lim_{x\to+\infty} x^{2n}\mathrm{e}^{-ax^2} - 2n\int_0^{+\infty} x^{2n-1}\mathrm{e}^{-ax^2}\,\mathrm{d}x\right) \\
&= \frac{n}{a}\int_0^{+\infty} x^{2n-1}\mathrm{e}^{-ax^2}\,\mathrm{d}x \\
&= \frac{n}{a}\,J_{n-1},
\end{aligned}
$$

or

$$\frac{J_n}{J_{n-1}} = \frac{n}{a}, \tag{2.35}$$

for all positive integers n. Note that, since a is positive, $\lim_{x\to+\infty} x^{2n}\mathrm{e}^{-ax^2} = 0$. This can be seen by the change of variable $x^2 = X$ and using l'Hopital's rule n times.

By Equation (2.35), one can see

$$\frac{J_1}{J_0} = \frac{1}{a}\ ,\ \frac{J_2}{J_1} = \frac{2}{a}\ ,\ \frac{J_3}{J_2} = \frac{3}{a}\ ,\ \dots\ ,\ \frac{J_n}{J_{n-1}} = \frac{n}{a}\cdot$$

Multiplying corresponding sides of these equations by each other and doing the cancelations, we get

$$\frac{J_n}{J_0} = \frac{1\cdot 2\cdot 3\cdots n}{a^n},$$

or

$$J_n = \frac{n!}{a^n}\,J_0. \tag{2.36}$$

But,

$$J_0 = \int_0^{+\infty} x\mathrm{e}^{-ax^2}\,\mathrm{d}x,$$

which is the same integral as in Example 2.3, except that the dummy variable r is now replaced by x, so $J_0 = \frac{1}{2a}$ and by Equation (2.36), $J_n = \frac{n!}{2a^{n+1}}$. ■

Note 2.14. *Of course, since this time $x^{2n+1}\mathrm{e}^{-ax^2}$ is an odd function,*

$$\int_{-\infty}^{+\infty} x^{2n+1}\mathrm{e}^{-ax^2}\,\mathrm{d}x = 0. \tag{2.37}$$

Note 2.15. *Equations (2.28), (2.32), and (2.34) are also true, if a is a complex number whose real part is positive.*

Example 2.15. Determine the value of $J = \int_{-\infty}^{+\infty} x^2 \mathrm{e}^{-\frac{(x-1)^2}{4}}\,\mathrm{d}x$.

Solution. Let $X = x - 1$, so $\mathrm{d}X = \mathrm{d}x$, $x = X + 1$, and as x goes from $-\infty$ to $+\infty$, so does X. Thus

$$\begin{aligned}
J &= \int_{-\infty}^{+\infty} (X+1)^2 \mathrm{e}^{-\frac{1}{4}X^2}\,\mathrm{d}X \\
&= \int_{-\infty}^{+\infty} (X^2 + 2X + 1)\mathrm{e}^{-\frac{1}{4}X^2}\,\mathrm{d}X \\
&= \underbrace{\int_{-\infty}^{+\infty} X^2 \mathrm{e}^{-\frac{1}{4}X^2}\,\mathrm{d}X}_{\text{Equation (2.33)}} + 2\underbrace{\int_{-\infty}^{+\infty} X \mathrm{e}^{-\frac{1}{4}X^2}\,\mathrm{d}X}_{\text{Equation (2.37)}} + \underbrace{\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{4}X^2}\,\mathrm{d}X}_{\text{Equation (2.27)}} \\
&= 6\sqrt{\pi}.
\end{aligned}$$

◇

Theorem 2.12. *Let a be a positive real and b be a real number. Then,*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2 - bx}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}}. \tag{2.38}$$

Proof. It is easy to see that

$$-ax^2 - bx = -a\left(x + \frac{b}{2a}\right)^2 + \frac{b^2}{4a},$$

so

$$\mathrm{e}^{-ax^2 - bx} = \mathrm{e}^{\frac{b^2}{4a}}\,\mathrm{e}^{-a\left(x + \frac{b}{2a}\right)^2}.$$

Therefore,

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2 - bx}\,\mathrm{d}x = \mathrm{e}^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} \mathrm{e}^{-a\left(x + \frac{b}{2a}\right)^2}\,\mathrm{d}x,$$

which by change of variable $X = x + \frac{b}{2a}$ can be written as

$$\mathrm{e}^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} \mathrm{e}^{-aX^2}\,\mathrm{d}X = \mathrm{e}^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}.$$

■

Note 2.16. *Equation (2.38) is also valid if a and b are complex numbers and* $\mathrm{Re}(a) > 0$. *In this case, $\sqrt{a}$ refers to the principal square root of a.*

Example 2.16. Determine the following integral,

$$J = \int_{-\infty}^{+\infty} e^{-[(1+\mathrm{i})x^2+\mathrm{i}\sqrt{\pi}x]}\,\mathrm{d}x.$$

Solution. This can be done by letting $a = 1+\mathrm{i}$ and $b = \mathrm{i}\sqrt{\pi}$ in Theorem 2.12. So

$$J = \sqrt{\frac{\pi}{1+\mathrm{i}}}\mathrm{e}^{\frac{(\mathrm{i}\sqrt{\pi})^2}{4(1+\mathrm{i})}} = \frac{\sqrt{\pi}}{\sqrt{1+\mathrm{i}}}\mathrm{e}^{\frac{-\pi}{4(1+\mathrm{i})}}. \tag{2.39}$$

Of course you can simplify this result more. I do it for the interested reader. First note that

$$\begin{aligned}\frac{-\pi}{4(1+\mathrm{i})} &= \frac{-\pi}{4(1+\mathrm{i})} \times \frac{1-\mathrm{i}}{1-\mathrm{i}} \\ &= -\frac{\pi(1-\mathrm{i})}{8} \\ &= -\frac{\pi}{8} + \frac{\pi}{8}\mathrm{i},\end{aligned}$$

so

$$\mathrm{e}^{\frac{-\pi}{4(1+\mathrm{i})}} = \mathrm{e}^{-\frac{\pi}{8}}\,\mathrm{e}^{\frac{\pi}{8}\mathrm{i}}.$$

On the other hand, according to Note 2.16, $\sqrt{1+\mathrm{i}}$ refers to the principal square root of $1+\mathrm{i}$. But $|1+i| = \sqrt{2}$ and $\mathrm{Arg}(1+\mathrm{i}) = \frac{\pi}{4}$ so

$$\sqrt{1+\mathrm{i}} = \sqrt[4]{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{8}}.$$

Plugging all these back into Equation (2.39), we get

$$J = \frac{\sqrt{\pi}}{\sqrt[4]{2}}\,\mathrm{e}^{-\frac{\pi}{8}}.$$

◇

Chapter 3

Fourier Analysis

This section contains some flavor of Fourier analysis, namely, Fourier series and Fourier integrals.

3.1 Fourier Series

Fourier series or Fourier expansion is applicable to periodic functions. So let us begin by recalling what a periodic function is.

Definition 3.1. A function f is said to be *periodic* if there exists a real number T such that for all x in the domain of f, $x+T$ also belongs to the domain of f and $f(x+T) = f(x)$. In this case, T is called a *period* of f. The smallest positive period of f is called *the fundamental* period of f.

Example 3.1. The functions $f(x) = \sin x$ and $g(x) = \tan \pi x$ are periodic functions, since

$$\begin{aligned} f(x+2\pi) &= \sin(x+2\pi) \\ &= \sin x \\ &= f(x), \end{aligned}$$

and

$$\begin{aligned} g(x+1) &= \tan \pi(x+1) \\ &= \tan(\pi x + \pi) \\ &= \tan \pi x \\ &= g(x). \end{aligned}$$

It is easy to see that, in fact, 2π and 1 are the fundamental periods of these functions, respectively.

Theorem 3.1. *Let f be a bounded*[§] *and piecewise continuous*[¶] *periodic function of fundamental period T such that it has a finite number of maxima and minima in each period*[‡]*. Then the trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=0}^{+\infty}\left[a_n \cos\left(\frac{2n\pi}{T}x_0\right) + b_n \sin\left(\frac{2n\pi}{T}x_0\right)\right], \tag{3.1}$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2n\pi}{T}x\right)\,\mathrm{d}x, \quad n = 0, 1, 2, 3, \ldots \tag{3.2}$$

and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2n\pi}{T}x\right)\,\mathrm{d}x, \quad n = 1, 2, 3, \ldots. \tag{3.3}$$

converges to $f(x_0)$ if f is continuous at x_0 and converges to

$$\frac{\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)}{2}, \tag{3.4}$$

if f is not continuous at x_0.

Note 3.1. *From Equation (3.2), it is seen that*

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\,\mathrm{d}x. \tag{3.5}$$

Therefore, almost always, one should treat a_0 and other a_n's separately, since the integrands for $n = 0$ and $n \neq 0$ cases, although expressed by the same formula, are different. The former does not include a cosine term while the latter does.

[§] A function f is said to be *bounded* if there exists a positive real number M such that for all x in the domain of f, $|f(x)| \leq M$.

[¶] A function f is said to be piecewise continuous on an interval $[a, b]$ if the interval can be partitioned by a finite number of points, $a = x_0 < x_1 < \cdots < x_n = b$ so that

1. f is continuous on each open subinterval (x_{i-1}, x_i).
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the interval.

[‡] One should not be concerned with these abstract mathematical conditions. These are always assumed to be the case in this course.

Definition 3.2. The trigonometric series in Equation (3.1) is called the *Fourier series* of f at $x = x_0$.

Example 3.2. Let f be a periodic function with period 2 which, in one period, is defined as follows,

$$f(x) = \begin{cases} 0, & -1 < x \leq 0 \\ x, & 0 \leq x < 1. \end{cases} \tag{3.6}$$

1. Determine the Fourier series of f at an arbitrary point x_0.

2. Use the Fourier series found in part (1) to determine the value of the following infinite series:

$$\sum_{n=1,3,5,\ldots}^{+\infty} \frac{1}{n^2}, \tag{3.7}$$

   by letting x_0 to be zero.

3. Answer the second part again, this time let $x_0 = 1$.

Solution.

1. As is mentioned in the example, $T = 2$. Hence, using Equations (3.5), (3.2), and (3.3) one can determine all the coefficients a_n and b_n as follows

$$\begin{aligned} a_0 &= \int_{-1}^{1} f(x)\,\mathrm{d}x \\ &= \int_{-1}^{0} 0\,\mathrm{d}x + \int_{0}^{1} x\,\mathrm{d}x \\ &= 0 + \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

   For all positive integers n,

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,\mathrm{d}x \\ &= \underbrace{\int_{-1}^{0} 0\cos(n\pi x)\,\mathrm{d}x}_{0} + \underbrace{\int_{0}^{1} x\cos(n\pi x)\,\mathrm{d}x}_{\text{Integration by parts}} \end{aligned}$$

$$= x\left[\frac{1}{n\pi}\sin(n\pi x)\right]_0^1 - 1\left[-\frac{1}{n^2\pi^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{1}{n^2\pi^2}[\cos(n\pi) - 1]$$

$$= \frac{1}{n^2\pi^2}[(-1)^n - 1],$$

and

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,\mathrm{d}x$$

$$= \underbrace{\int_{-1}^{0} 0\sin(n\pi x)\,\mathrm{d}x}_{0} + \underbrace{\int_{0}^{1} x\sin(n\pi x)\,\mathrm{d}x}_{\text{Integration by parts}}$$

$$= x\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^1 - 1\left[-\frac{1}{n^2\pi^2}\sin(n\pi x)\right]_0^1$$

$$= \frac{(-1)^{n+1}}{n\pi}\cdot$$

Plugging these into Equation (3.1), for an arbitrary point x_0, one ends with the following Fourier series for the function f,

$$\frac{1}{4} + \sum_{n=1}^{+\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x_0) + \frac{(-1)^{n+1}}{n\pi}\sin(n\pi x_0)\right]. \tag{3.8}$$

2. Since the function f is continuous at $x_0 = 0$, according to Theorem 3.1, the series (3.8) converges to $f(0)$. Therefore,

$$f(0) = \frac{1}{4} + \sum_{n=1}^{+\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi 0) + \frac{(-1)^{n+1}}{n\pi}\sin(n\pi 0)\right], \tag{3.9}$$

or, since $\cos 0 = 1$, $\sin 0 = 0$, and $f(0) = 0$,

$$0 = \frac{1}{4} + \sum_{n=1}^{+\infty}\frac{(-1)^n - 1}{n^2\pi^2}\cdot \tag{3.10}$$

But $(-1)^n - 1$ is equal to zero if n is even and is equal to -2 if n is odd. Therefore,

$$0 = \frac{1}{4} - \frac{2}{\pi^2}\sum_{n=1,3,5,\ldots}^{+\infty}\frac{1}{n^2},$$

or

$$\sum_{n=1,3,5,\ldots}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

3. This time, f is no longer continuous at the given point $x_0 = 1$, since

$$\lim_{x \to 1^-} f(x) = 1, \tag{3.11}$$

and

$$\lim_{x \to 1^+} f(x) = 0, \tag{3.12}$$

and these limits are not equal. Hence, the Fourier series of this function at $x_0 = 1$ does not converge to $f(1)$ but, according to Equation (3.4), and making use of Equations (3.11) and (3.12), it converges to

$$\frac{1+0}{2} = \frac{1}{2}.$$

So

$$\frac{1}{2} = \frac{1}{4} + \sum_{n=1}^{+\infty} \left[\frac{(-1)^n - 1}{n^2\pi^2} \cos(n\pi 1) + \frac{(-1)^{n+1}}{n\pi} \sin(n\pi 1) \right]. \tag{3.13}$$

But $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, Equation (3.13) yields

$$\frac{1}{2} = \frac{1}{4} + \sum_{n=1}^{+\infty} \left[\frac{(-1)^n - 1}{n^2\pi^2} (-1)^n \right], \tag{3.14}$$

or

$$\frac{1}{2} - \frac{1}{4} = \frac{1}{\pi^2} \sum_{n=1}^{+\infty} \frac{1 - (-1)^n}{n^2},$$

or

$$\frac{1}{4} = \frac{2}{\pi^2} \sum_{n=1,3,5,\ldots}^{+\infty} \frac{1}{n^2},$$

form which the same result as in part 2 follows.

◇

Note 3.2. *Let f be a function which fulfills the conditions of Theorem 3.1. Then if f is an odd function, so is $f(x)\cos\left(\frac{2n\pi}{T}x\right)$, and according to Theorem 2.5, all coefficients a_n would be zero. Therefore, the Fourier series of an odd function f would be*

$$\sum_{n=1}^{+\infty} b_n \sin\left(\frac{2n\pi}{T}x\right). \tag{3.15}$$

But, since in this case $f(x)\sin\left(\frac{2n\pi}{T}x\right)$ is an even function, Equation (3.3) reduces to

$$b_n = \frac{4}{T}\int_0^{T/2} f(x)\sin\left(\frac{2n\pi}{T}x\right)\,\mathrm{d}x, \quad n = 1, 2, 3, \ldots. \tag{3.16}$$

Similarly, if f is an even function, b_n's would be zero and the Fourier series of f takes the following form

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{2n\pi}{T}x\right), \tag{3.17}$$

where

$$a_n = \frac{4}{T}\int_0^{T/2} f(x)\sin\left(\frac{2n\pi}{T}x\right)\,\mathrm{d}x, \quad n = 0, 1, 2, 3, \ldots. \tag{3.18}$$

Example 3.3. Let f be a periodic function of period 2π which on the interval $[-\pi, \pi]$ is defined as follows:

$$f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ 0, & x = 0, \\ 1, & 0 < x \le \pi. \end{cases}$$

Determine its Fourier series at an arbitrary point x.

Solution. Since this function is an odd function, based on the Note 3.2, its Fourier series is

$$\sum_{n=1}^{+\infty} b_n \sin(nx),$$

where

$$b_n = \frac{4}{2\pi}\int_0^{\pi} f(x)\sin\left(\frac{2n\pi}{2\pi}x\right)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{\pi} 1\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} \\
&= -\frac{2}{n\pi}[-\cos(n\pi)+1] \\
&= \frac{2}{n\pi}\,[1-(-1)^n].
\end{aligned}
$$

Therefore, the Fourier series of this function is

$$\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{1-(-1)^n}{n}\,\sin(nx).$$

Since

$$1-(-1)^n=\begin{cases}0, & n \text{ is even,}\\ 2, & n \text{ is odd,}\end{cases}$$

this Fourier series can also be written as

$$\frac{4}{\pi}\sum_{k=1}^{+\infty}\frac{1}{2k-1}\sin(2k-1)x.$$

$\diamond$

Note 3.3. *Fourier series of a function is unique. Therefore, if by any method other than the standard method described above, one can come to an expansion in the form of Equation (3.1) for f, one should be certain that one has the Fourier series expansion of the function f.*

Example 3.4. Find the Fourier expansion of the function $f(x)=\sin^3 x$.

Solution. Of course, one can use the standard procedure to find the Fourier series. But let us make use of Note 3.3 this time. In Example 1.6, we found

$$\sin 3x = 3\sin x - 4\sin^3 x,$$

so

$$\sin^3 x = \frac{3}{4}\sin x + \left(-\frac{1}{4}\right)\sin 3x. \tag{3.19}$$

Since f is an odd function of period 2π, its Fourier series is

$$\sum_{n=1}^{+\infty} b_n\sin(nx) = b_1\sin x + b_2\sin(2x) + b_3\sin(3x) + b_4\sin(4x) + \cdots. \tag{3.20}$$

Comparing Equations (3.19) and (3.20), since the Fourier series of a function is unique, it is clear that Equation (3.19) is itself the Fourier series of f and in fact, $b_1 = \frac{3}{4}$, $b_3 = -\frac{1}{4}$, and all other coefficients are zero. ◇

Before ending up with this section, let us look at what is, so-called, the complex form of the Fourier series of a function.

Let f be a function with properties stated in Theorem 3.1, and consider the expression (3.1) with x_0 replaced by x and let $\omega_n := \frac{2n\pi}{T}$, so $\omega_{-n} = -\omega_n$. Our goal is to write this expression as a sum of complex exponential functions with appropriate coefficients and to find a formula enabling us to determine these coefficients. Using Equations (1.21), this expression can be written as follows:

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \frac{\mathrm{e}^{\mathrm{i}\omega_n x} + \mathrm{e}^{-\mathrm{i}\omega_n x}}{2} + b_n \frac{\mathrm{e}^{\mathrm{i}\omega_n x} - \mathrm{e}^{-\mathrm{i}\omega_n x}}{2\mathrm{i}} \right) \tag{3.21}$$

or

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(\frac{a_n - \mathrm{i}b_n}{2} \mathrm{e}^{\mathrm{i}\omega_n x} + \frac{a_n + \mathrm{i}b_n}{2} \mathrm{e}^{-\mathrm{i}\omega_n x} \right). \tag{3.22}$$

Letting

$$\frac{a_0}{2} = c_0,$$

and

$$\frac{a_n - \mathrm{i}b_n}{2} = c_n, \tag{3.23}$$

expression (3.22) reduces to

$$c_0 + \sum_{n=1}^{+\infty} \left(c_n \mathrm{e}^{\mathrm{i}\omega_n x} + c_n^* \mathrm{e}^{-\mathrm{i}\omega_n x} \right). \tag{3.24}$$

Using Equations (3.2) and (3.3), in Equation (3.23), one gets

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-T/2}^{T/2} f(x)[\cos \omega_n x - \mathrm{i} \sin \omega_n x] \, \mathrm{d}x \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \mathrm{e}^{-\mathrm{i}\omega_n x} \, \mathrm{d}x, \end{aligned} \tag{3.25}$$

and, since f is assumed to be real,

$$c_n^* = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \mathrm{e}^{\mathrm{i}\omega_n x} \, \mathrm{d}x$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(x)\mathrm{e}^{-\mathrm{i}\omega_{-n}x}\,\mathrm{d}x$$
$$= c_{-n}.$$

Therefore, one can write the expression (3.24) as

$$c_0 + \sum_{n=1}^{+\infty}\left(c_n\mathrm{e}^{\mathrm{i}\omega_n x} + c_{-n}\mathrm{e}^{\mathrm{i}\omega_{-n}x}\right),$$

or

$$c_0 + \sum_{n=1}^{+\infty} c_n\mathrm{e}^{\mathrm{i}\omega_n x} + \sum_{n=1}^{+\infty} c_{-n}\mathrm{e}^{\mathrm{i}\omega_{-n}x},$$

or

$$\sum_{n=1}^{+\infty} c_n\mathrm{e}^{\mathrm{i}\omega_n x} + c_0 + \sum_{n=-\infty}^{-1} c_n\mathrm{e}^{\mathrm{i}\omega_n x},$$

or more compactly as

$$\sum_{n=-\infty}^{+\infty} c_n\mathrm{e}^{\mathrm{i}\omega_n x}. \tag{3.26}$$

This series, where $\omega_n = \frac{2n\pi}{T}$ and c_n's are determined by Equation (3.25), is called the *complex* Fourier series of f.

Example 3.5. Find the complex Fourier expansion of the function f in Example 3.3.

Solution. For this function $T = 2\pi$, so $\omega_n = n$ and

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-1)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x + \int_{0}^{\pi}(1)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{\mathrm{i}n}\mathrm{e}^{-\mathrm{i}nx}|_{-\pi}^{0} - \frac{1}{\mathrm{i}n}\mathrm{e}^{-\mathrm{i}nx}|_{0}^{\pi}\right) \\ &= \frac{2-(\mathrm{e}^{\mathrm{i}nx}+\mathrm{e}^{-\mathrm{i}nx})}{2n\pi\mathrm{i}} \\ &= \frac{1-\cos(nx)}{n\pi\mathrm{i}} \end{aligned}$$

$$= \frac{1-(-1)^n}{n\pi \mathrm{i}}.$$

Therefore, its complex Fourier series is

$$\sum_{n=-\infty}^{+\infty} \frac{1-(-1)^n}{n\pi \mathrm{i}} \mathrm{e}^{\mathrm{i}nx}.$$

◇

3.2 Fourier Transform

As we saw in the last section, Fourier series is useful in representing either periodic functions or functions confined in limited range of interest. However, in many problems this is not the case and we are dealing with a nonperiodic function over an infinite range. In such cases, we can imagine that the function is periodic with period approaching to infinity.

Consider a function f which is not necessarily periodic. Define a function f_T to be the periodic function of period T which is equal to f within the interval $[-T/2, T/2]$. As mentioned in the last section, f_T can be expanded in a complex Fourier series

$$f_T(x) = \sum_{n=-\infty}^{+\infty} c_n \mathrm{e}^{\mathrm{i}\omega_n x}, \tag{3.27}$$

where

$$\omega_n = \frac{2n\pi}{T}, \tag{3.28}$$

and

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(\xi) \mathrm{e}^{-\mathrm{i}\omega_n \xi} \, \mathrm{d}\xi. \tag{3.29}$$

When T approaches infinity, f_T becomes the same as f. We shall therefore let T approaches infinity in the expression above. From the definition of ω_n in Equation (3.28), we have

$$\frac{\omega_{n+1} - \omega_n}{2\pi} = \frac{1}{T}. \tag{3.30}$$

Substituting (3.29) and (3.30) in the Equation (3.27), we get

$$f_T(x) = \sum_{n=-\infty}^{+\infty} \left[\frac{\omega_{n+1} - \omega_n}{2\pi} \mathrm{e}^{\mathrm{i}\omega_n x} \int_{-T/2}^{T/2} f(\xi) \mathrm{e}^{-\mathrm{i}\omega_n \xi} \, \mathrm{d}\xi \right]. \tag{3.31}$$

Now we take the limit of both sides of the above equation when T goes to infinity. As mentioned earlier, the left hand side would be $f(x)$. For the right hand side, from Equation (3.30) we see that $\omega_{n+1}-\omega_n$ goes to zero. This means that the difference of two consecutive values of ω gets very small and essentially, ω_n can be replaced by a continuous variable ω and $\omega_{n+1}-\omega_n$ can be replaced by $\mathrm{d}\omega$. In this situation, the sum over n transforms to a definite integral over ω and since n goes from $-\infty$ to $+\infty$, from Equation (3.28), one can see that the same is true for ω. So we end up with

$$f(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{\mathrm{i}\omega x} \int_{-\infty}^{+\infty} f(\xi)\mathrm{e}^{-\mathrm{i}\omega\xi}\, \mathrm{d}\xi, \tag{3.32}$$

which can also be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi)\mathrm{e}^{-\mathrm{i}\omega\xi}\, \mathrm{d}\xi \right). \tag{3.33}$$

The expression in the brackets in Equation (3.33) is a function of ω which we denote it by $\tilde{f}(\omega)$. Therefore Equation (3.33) will be

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}\omega, \tag{3.34}$$

where

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi)\mathrm{e}^{-\mathrm{i}\omega\xi}\, \mathrm{d}\xi,$$

or after changing the dummy variable ξ to x,

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}x. \tag{3.35}$$

$\tilde{f}(\omega)$ in Equation (3.35) is called the *Fourier transform* of $f(x)$.

Example 3.6. Find the Fourier transform of the following function:

$$f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Solution. From Equation (3.35), the Fourier transform $\tilde{f}(\omega)$ of this function is

$$\begin{aligned}
\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-1} 0\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x + \int_{-1}^{1} 1\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x + \int_{1}^{+\infty} 0\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{\mathrm{i}\omega} \mathrm{e}^{-\mathrm{i}\omega x} \right]_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{\mathrm{i}\omega} - \mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}.
\end{aligned}$$

◇

Chapter 4

Dirac Delta and Step Functions

In this section we are going to study informally what is called the *delta function*, introduce the step function, and find out its relation to the delta function.

4.1 Dirac Delta "Function"

Let ε be a positive real number. Consider the family of functions $d_\varepsilon(x)$ depending on the parameter ε as follows,

$$d_\varepsilon(x) = \begin{cases} \dfrac{1}{2\varepsilon}, & |x| \leq \varepsilon \\ 0, & |x| > \varepsilon. \end{cases} \tag{4.1}$$

Sketching the graph of these functions for some small values of ε, you are easily convinced that as ε gets smaller, the nonzero part of the graph gets thinner and higher. But the area under these graphs are always constant and it is equal to 1, since for any $\varepsilon > 0$,

$$\begin{aligned} \text{Area} &= \int_{-\infty}^{+\infty} d_\varepsilon(x)\,\mathrm{d}x \\ &= \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,\mathrm{d}x \\ &= 1. \end{aligned}$$

Definition 4.1. Let x be a real number. The *Dirac delta function* at x is denoted by $\delta(x)$ and it is defined by

$$\delta(x) = \lim_{\varepsilon \to 0^+} d_\varepsilon(x). \tag{4.2}$$

Let us write the Equation (4.2) in a more concrete manner. Suppose that $x = 0$. Then, for all $\varepsilon > 0$, $|0| \leq \varepsilon$ and $d_\varepsilon(0) = \frac{1}{2\varepsilon}$. So

$$\begin{aligned}\delta(0) &= \lim_{\varepsilon \to 0^+} d_\varepsilon(0)\\ &= \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon}\\ &= +\infty.\end{aligned}$$

If $x \neq 0$, there exists an $\varepsilon_0 > 0$ so that $|x| > \varepsilon_0$. Since ε approaches zero from above, eventually, $\varepsilon < \varepsilon_0$. Hence, $|x| > \varepsilon$ and, consequently, $d_\varepsilon(x) = 0$ and

$$\begin{aligned}\delta(x) &= \lim_{\varepsilon \to 0^+} d_\varepsilon(x)\\ &= \lim_{\varepsilon \to 0^+} 0\\ &= 0.\end{aligned}$$

Therefore, Equation (4.2) can be replaced by the following one

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0. \end{cases} \tag{4.3}$$

Note 4.1. *The first thing that one should note about the delta function is that, in fact, it is not a function at all. Since, as you know, a function assigns a number to different numbers but delta function assigns $+\infty$ to zero and $+\infty$ is not a number. To be mathematically rigorous, Dirac delta function is a* distribution *function. At this level, to introduce the delta function more precisely, one can say that the delta function is a shorthand notation that is reminiscent of a limiting process so that using it considerably shortens the calculations.*

Let us now, informally, determine the area under the graph of $\delta(x)$.

$$\text{Area} = \int_{-\infty}^{+\infty} \delta(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \left(\lim_{\varepsilon \to 0^+} d_\varepsilon(x) \right) \mathrm{d}x \\
&= \lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} d_\varepsilon(x) \, \mathrm{d}x \\
&= \lim_{\varepsilon \to 0^+} 1 \\
&= 1.
\end{aligned}
$$

Therefore, Dirac delta function can also be introduced as follows

$$
\delta(x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}, \quad \int_{-\infty}^{+\infty} \delta(x) \, \mathrm{d}x = 1. \tag{4.4}
$$

From geometrical point of view, $\delta(x)$ can be imagined as a very sharp pike located on the y axis. Of course, this pike can be located everywhere, so one can similarly consider $\delta(x-a)$ as follows

$$
\delta(x-a) = \begin{cases} 0, & x \neq a \\ +\infty, & x = a \end{cases}, \quad \int_{-\infty}^{+\infty} \delta(x-a) \, \mathrm{d}x = 1. \tag{4.5}
$$

Note 4.2. *It is clear that the delta function is an even function, i.e., for any real value of x, $\delta(-x) = \delta(x)$. Of course, this also means that $\delta(x - x') = \delta(x' - x)$, for any real values of x and x'.*

Theorem 4.1. *For a function f*

1. *$f(x)\delta(x-a) = f(a)\delta(x-a)$,*
2. *$\int_{-\infty}^{+\infty} f(x)\delta(x-a) \, \mathrm{d}x = f(a)$,*
3. *If $\alpha < \beta$, then*

$$
\int_\alpha^\beta f(x)\delta(x-a) \, \mathrm{d}x = \begin{cases} f(a), & a \in (\alpha, \beta) \\ 0, & a \notin (\alpha, \beta). \end{cases}
$$

Example 4.1. Determine the value of the following integrals:

1. $I = \int_2^6 (3x^2 - 2x - 1) \, \delta(x-3) \, \mathrm{d}x$,

2. $J = \int_{-1}^{1} x^2\,\delta(3x+1)\,\mathrm{d}x$,

3. $K = \int_{1}^{5} \cos(x^2)\,\delta(x+4)\,\mathrm{d}x$.

Solution.

1. Let $f(x) = 3x^2 - 2x - 1$. Since 3 belongs to the interval $(2, 6)$, by the last part of Theorem 4.1,

$$\begin{aligned} I &= f(3) \\ &= 3 \times 3^2 - 2 \times 3 - 1 \\ &= 20. \end{aligned}$$

2. Let $3x = X$, so $\mathrm{d}x = \frac{1}{3}\,\mathrm{d}X$ and since x belongs to the interval $(-1, 1)$, X belongs to the interval $(-3, 3)$. Thus

$$\begin{aligned} J &= \int_{-3}^{3} \left(\frac{X}{3}\right)^2 \delta(X+1)\,\mathrm{d}X \\ &= \frac{1}{9}\int_{-3}^{3} X^2 \delta(X-(-1))\,\mathrm{d}X \\ &= \frac{1}{9}\cdot(-1)^2 \\ &= \frac{1}{9}. \end{aligned}$$

In the last step, I used the third part of Theorem 4.1.

3. K can be thought as

$$K = \int_{1}^{5} \cos(x^2)\,\delta(x-(-4))\,\mathrm{d}x.$$

But this time the interval of integration $(1, 5)$ does not contain -4 and therefore by the last part of Theorem 4.1, $K = 0$.

◇

Example 4.2. Determine the Fourier transform of $f(x) = \delta(x-a)$.

Solution. Using Equation(3.35) and the second part of Theorem 4.1, one finds

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \delta(x-a)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}\omega a}.$$

◇

Theorem 4.2. *The delta function $\delta(x - x')$ can be represented as follows*

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega(x-x')} \, \mathrm{d}\omega. \tag{4.6}$$

Proof. Since the integration variable is ω, the right hand side of Equation (4.6) is a function of x and x' which we call it $G(x, x')$. Changing the dummy integration variable from x to x' in Equation (3.35) and plugging it in Equation (3.34), one gets

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x') \mathrm{e}^{-\mathrm{i}\omega x'} \, \mathrm{d}x' \right) \mathrm{e}^{\mathrm{i}\omega x} \, \mathrm{d}\omega \\ &= \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x') \mathrm{e}^{\mathrm{i}\omega(x-x')} \, \mathrm{d}x' \right) \mathrm{d}\omega \\ &= \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x') \mathrm{e}^{\mathrm{i}\omega(x-x')} \, \mathrm{d}\omega \right) \mathrm{d}x' \\ &= \int_{-\infty}^{+\infty} f(x') \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega(x'-x)} \, \mathrm{d}\omega \right) \mathrm{d}x'. \end{aligned}$$

In the second step above, $\mathrm{e}^{\mathrm{i}\omega x}$ slipped into the integral. This is possible since $\mathrm{e}^{\mathrm{i}\omega x}$ does not depend on the integration variable x'. In the third step, the order of integration is changed. Therefore,

$$f(x) = \int_{-\infty}^{+\infty} f(x') G(x, x') \, \mathrm{d}x. \tag{4.7}$$

Comparing Equation (4.7) with the second part of Theorem 4.1, yields the Equation (4.6). ■

Note 4.3. *By a change of variable from ω to $-\omega$, in Equation (4.6), $\delta(x - x')$ can also be written as*

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\omega(x-x')} \, \mathrm{d}\omega. \tag{4.8}$$

Definition 4.2. Two functions $D_1(x)$ and $D_2(x)$ involving the delta functions are said to be *equal* if for any ordinary function[§]

$$\int_{-\infty}^{+\infty} f(x)D_1(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)D_2(x)\,\mathrm{d}x. \tag{4.9}$$

Example 4.3. Let c be a non-zero real constant. Show that

$$\delta(cx) = \frac{1}{|c|}\delta(x). \tag{4.10}$$

Solution. Let $f(x)$ be an ordinary function and let $D_1(x) = \delta(cx)$ and $D_2(x) = \frac{1}{|c|}\delta(x)$. We show that Equation (4.9) holds. The right hand side of Equation (4.9) is

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)D_2(x)\,\mathrm{d}x &= \int_{-\infty}^{+\infty} f(x)\left(\frac{1}{|c|}\delta(x)\right)\mathrm{d}x \\ &= \frac{1}{|c|}\int_{-\infty}^{+\infty} f(x)\delta(x)\,\mathrm{d}x \\ &= \frac{1}{|c|}f(0), \end{aligned} \tag{4.11}$$

where in the last step, the second part of Theorem 4.1 is used. For the left hand side of Equation (4.9), one should consider $c < 0$ and $c > 0$ separately. First, let $c < 0$ and apply the change of variable $X = cx$. Therefore, $\mathrm{d}x = \frac{1}{c}\mathrm{d}X$ and $X|_{+\infty}^{-\infty}$ as $x|_{-\infty}^{+\infty}$. So

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)D_1(x)\,\mathrm{d}x &= \int_{-\infty}^{+\infty} f(x)\delta(cx)\,\mathrm{d}x \\ &= \int_{+\infty}^{-\infty} f\left(\frac{X}{c}\right)\delta(X)\frac{1}{c}\,\mathrm{d}X \\ &= -\frac{1}{c}\int_{-\infty}^{+\infty} f\left(\frac{X}{c}\right)\delta(X)\,\mathrm{d}X \\ &= \frac{1}{|c|}f(0). \end{aligned} \tag{4.12}$$

The validity of Equation (4.9) is clear by comparing Equations (4.11) and (4.12). Similarly, one can show that the Equation (4.9) is also true for the case of positive c. ◇

[§]A function not involving the delta function.

4.2 Step Function

Definition 4.3. The fuction $\theta(x)$ defined as

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0, \end{cases} \tag{4.13}$$

is called the *step* function.

Theorem 4.3. *For a non-zero x,*

$$\frac{\mathrm{d}}{\mathrm{d}x}|x| = 2\theta(x) - 1. \tag{4.14}$$

Proof. By Definition 4.3,

$$\begin{aligned} 2\theta(x) - 1 &= 2 \times \left(\begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \right) - 1 \\ &= \begin{cases} 2 \times 1 - 1, & x > 0 \\ 2 \times 0 - 1, & x < 0 \end{cases} \\ &= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}|x|. \end{aligned}$$

■

Note 4.4. *From what is calculated for $2\theta(x) - 1$ in the proof above, it is readily seen that for $x \neq 0$*

$$(2\theta(x) - 1)^2 = 1. \tag{4.15}$$

Theorem 4.4. *The derivative of the step function is the delta function, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta(x) = \delta(x). \tag{4.16}$$

Proof. Let $D_1(x) = \delta(x)$ and $D_2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\theta(x)$. From the second part of Theorem 4.1 can be deduced that the left hand side of Equation (4.9) is $f(0)$. We show that the right hand side (R.H.S.) of the Equation (4.9) is also $f(0)$.

$$\begin{aligned}
\text{R.H.S.} &= \underbrace{\int_{-\infty}^{+\infty} f(x) \frac{\mathrm{d}}{\mathrm{d}x}\theta(x)\,\mathrm{d}x}_{\text{integration by parts}} \\
&= \lim_{x\to+\infty} f(x)\theta(x) - \lim_{x\to-\infty} f(x)\theta(x) - \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}x} f(x)\theta(x)\,\mathrm{d}x \\
&= \lim_{x\to+\infty} f(x) - 0 - \int_{0}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}x} f(x)\,\mathrm{d}x \\
&= \lim_{x\to+\infty} f(x) - \left(\lim_{x\to+\infty} f(x) - f(0)\right) \\
&= f(0).
\end{aligned}$$

Therefore, Equation (4.9) holds. ■

Example 4.4. Let λ be a real parameter. Determine the second derivative of the function $f(x) = \mathrm{e}^{\lambda|x|}$ for an arbitrary x.

Solution. The first derivative is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\,(\lambda|x|)\,\mathrm{e}^{\lambda|x|} \\
&= \lambda \frac{\mathrm{d}}{\mathrm{d}x}(|x|)\mathrm{e}^{\lambda|x|} \\
&= \lambda(2\theta(x) - 1)\mathrm{e}^{\lambda|x|} \\
&= \lambda(2\theta(x) - 1)f(x).
\end{aligned}$$

Therefore, the second derivative would be

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{e}^{\lambda|x|} &= \lambda\left[\frac{\mathrm{d}}{\mathrm{d}x}(2\theta(x) - 1)\right] f(x) + \lambda(2\theta(x) - 1)\frac{\mathrm{d}}{\mathrm{d}x} f(x) \\
&= 2\lambda\delta(x)f(x) + \lambda^2(2\theta(x) - 1)^2 f(x) \\
&= 2\lambda\delta(x)f(0) + \lambda^2 f(x) \\
&= 2\lambda\delta(x) + \lambda^2\mathrm{e}^{\lambda|x|}.
\end{aligned}$$

In the calculations above I used the first part of Theorem 4.1 and also Note 4.4. ◇

Chapter 5

Differential Equations

In this section I will go through the main concepts of ordinary and partial differential equations.

5.1 Ordinary Differential Equations

First ordinary differential equations, the ones containing the ordinary derivatives.

5.1.1 Fundamental Concepts

Definition 5.1. An equation containing an unknown function in one variable and its derivatives is called an *ordinary differential equation* or *ODE* for short.

In an ODE, the order of the derivative with the highest order is said to be the *order* of the ODE.

Example 5.1. The following equations are examples of ODE's,

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = 5x + 3$,

2. $\mathrm{e}^y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = 1$,

3. $(y''')^3 + 3y(y')^7 + y^3(y')^2 = 5x$.

The order of these equations are 1, 2, and 3, respectively.

Note 5.1. *The most general form of an ODE of order n is*

$$F(x; y, y', y'', \dots, y^{(n)}) = 0, \tag{5.1}$$

where F is a function of $n+2$ variables.

Definition 5.2. A function $y = \phi(x)$ is said to be a *solution* of the Equation (5.1) on the interval (α, β), if for any x_0 of this interval

$$F\left(x_0; \phi(x_0), \phi'(x_0), \phi''(x_0), \ldots, \phi^{(n)}(x_0)\right) = 0. \tag{5.2}$$

Note 5.2. *In Definition 5.2, α and/or β can be chosen at infinity.*

Note 5.3. *Clearly, if $y = \phi(x)$ is a solution of a ODE of order n on an interval (α, β), then it should be differentiable up to order n on this interval and therefore, all its derivatives up to order $n-1$ must be continuous on (α, β).*

Example 5.2. Show that the function $y = \frac{1}{x^2-1}$ is a solution of $y' + 2xy^2 = 0$ on the interval $(-1, 1)$. Is this function also a solution for the same equation on the interval $(-2, 2)$?

Solution. Plugging this given function into the right hand side of the given equation, for all x in the interval $(-1, 1)$, we have

$$\begin{aligned}
\text{L.H.S.} &= \left(\frac{1}{x^2-1}\right)' + 2x\left(\frac{1}{x^2-1}\right)^2 \\
&= \frac{-2x}{(x^2-1)^2} + \frac{2x}{(x^2-1)^2} \\
&= 0 \\
&= \text{R.H.S.}.
\end{aligned}$$

But, since the given function is not differentiable at $x = \pm 1$, this function is not the solution of this equation on the interval $(-2, 2)$ which contains these points. $\diamond$

Definition 5.3. Let $y = \phi(x)$ be a solution of an ODE of order n on the interval I. If $\phi(x)$ consists of n arbitrary constant parameters, then $y = \phi(x)$ is said to be the *general* solution of the given ODE.

Example 5.3. Show that $y = a \sin 2x + b \cos 2x$, where a and b are arbitrary constant parameters, is the general solution of $y'' + 4y = 0$ on the whole real line.

Solution. We have

$$\begin{aligned} y'' + 4y &= (a\sin 2x + b\cos 2x)'' + 4(a\sin 2x + b\cos 2x) \\ &= (2a\cos 2x - 2b\sin 2x)' + 4(a\sin 2x + b\cos 2x) \\ &= -4a\sin 2x - 2b\cos 2x + 4a\sin 2x + 4b\cos 2x \\ &= 0. \end{aligned}$$

◇

Example 5.4. Show for arbitrary constant parameters C and D,

$$u(\rho) = C\rho^{l+1} + D\rho^{-l},$$

is the general solution of the following equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} = \frac{l(l+1)}{\rho^2}u,$$

on the interval $(0, +\infty)$.

Solution. For all ρ in the interval $(0, +\infty)$, we have

$$\begin{aligned} \frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} &= \frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left(C\rho^{l+1} + D\rho^{-l}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}\rho}\left(C(l+1)\rho^{l} - Dl\rho^{-l-1}\right) \\ &= Cl(l+1)\rho^{l-1} - Dl(-l-1)\rho^{-l-2} \\ &= \frac{l(l+1)}{\rho^2}\left(C\rho^{l+1} + D\rho^{-l}\right) \\ &= \frac{l(l+1)}{\rho^2}u\cdot \end{aligned}$$

◇

Example 5.5. Show that $\psi(\xi) = A\mathrm{e}^{-\frac{\xi^2}{2}} + B\mathrm{e}^{\frac{\xi^2}{2}}$, where A and B are arbitrary constant parameters, can be considered as the general solution of the following equation:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} = \xi^2\psi,$$

on $\mathbb{R}$ for very large values of ξ.

Solution. We have

$$\frac{\mathrm{d}\psi}{\mathrm{d}\xi} = -A\xi \mathrm{e}^{-\frac{\xi^2}{2}} + B\xi \mathrm{e}^{\frac{\xi^2}{2}},$$

and

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} = -A\mathrm{e}^{-\frac{\xi^2}{2}} + A\xi^2\mathrm{e}^{-\frac{\xi^2}{2}} + B\mathrm{e}^{\frac{\xi^2}{2}} + B\xi^2\mathrm{e}^{\frac{\xi^2}{2}}.$$

But for very large values of ξ, $\mathrm{e}^{-\frac{\xi^2}{2}}$ is negligible and can be ignored and $\mathrm{e}^{\frac{\xi^2}{2}}$ is very small compared to $\xi^2\mathrm{e}^{\frac{\xi^2}{2}}$ and it can also be ignored. Therefore,

$$\begin{aligned}\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} &= \xi^2(A\mathrm{e}^{-\frac{\xi^2}{2}} + B\mathrm{e}^{\frac{\xi^2}{2}}) \\ &= \xi^2\psi.\end{aligned}$$

◇

Definition 5.4. A solution of an ODE which is obtained by assigning specific values to the constant parameters of its general solution is called a *particular* solution of the given ODE.

Example 5.6. By assigning 5 to a and -3 to b in Example 5.3, one ends up with $y = 5\sin 2x - 3\cos 2x$ which is a particular solution of $y'' + 4y = 0$. Clearly, this equation has infinite number of particular solutions.

Note 5.4. *It happens that there might be a solution of an ODE which cannot be obtained by assigning any particular value to the constant parameters of the general solution of the ODE. This kind of solutions are called* singular *solutions of the ODE.*

Example 5.7. Cosider the equation $(y')^2 - xy' + y = 0$.

1. Show that $y = cx - c^2$, where c is an arbitrary constant parameter, is the general solution of the given equation.

2. Show that $y = \frac{1}{2}x - \frac{1}{4}$ is a particular solution of this equation.

3. Is $y = \frac{1}{4}x^2$ a solution of this equation? Can it be obtained from its general solution by assigning an appropriate value to c?

Solution.

1. Since $y' = c$, we have

$$\begin{aligned}(y')^2 - xy' + y &= c^2 - cx + cx - c^2\\ &= 0,\end{aligned}$$

   so $y = cx - x^2$ is a solution of this equation and since the given ODE is of degree one and this solution depends on only one parameter c, $y = cx - c^2$ is the general solution of the given ODE.

2. Plugging $c = \frac{1}{2}$ in the general solution given in part 1, one gets $y = \frac{1}{2}x - \frac{1}{4}$. Therefore, $y = \frac{1}{2}x - \frac{1}{4}$ is a particular solution of this ODE.

3. $y' = \frac{1}{2}x$, so

$$\begin{aligned}(y')^2 - xy' + y &= \frac{1}{4}x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2\\ &= 0.\end{aligned}$$

   Hence, $y = \frac{1}{4}x^2$ is also a solution of this equation. But, obviously, this solution cannot be obtained from the general solution in part 1, by assigning a particular value to the parameter c, so $y = \frac{1}{4}x^2$ is a singular solution of this ODE.

$\diamond$

Definition 5.5. An ODE together with complementary conditions on the unknown function and its derivatives, all expressed for the same value of the independent variable of the equation, is said to be an *initial value* problem. On the other hand, if these complementary conditions are expressed at more than one value for the independent variable of the equation, one is dealing with a, so-called, *boundary value* problem.

Example 5.8. The equation $y'' + 2y' = \mathrm{e}^x$ together with complementary conditions $y(\pi) = 1$ and $y'(\pi) = 2$ is an initial value problem, since both conditions are expressed at the same value π for x. But the same equation along with the complementary conditions, $y(0) = 3$ and $y(1) = 4$, or, $y(0) = 1$ and $y'(2) = 1$, is a boundary value problem, since in the former, the conditions are expressed at different values 0 and 1 for x and in the latter, they are expressed at different values 0 and 2 for x.

First Order Ordinary Differential Equations

The general form of a first order ODE is

$$F(x; y, y') = 0. \tag{5.3}$$

Since there is no general method to solve all this kind of ODE's, we categorize them to more simple ones, but before that we assume that Equation (5.3) can always be solved for y', i.e., there exists a function f in two variables such that

$$y' = f(x; y). \tag{5.4}$$

Definition 5.6. If a first order ODE can be reduced to the form in Equation (5.4), where f is a function only of x or a funciton only of y, then the ODE is called a *separable* ODE.

To solve a separable ODE, if f is a function of only x, then Equation (5.4) reduces to

$$y' = f(x),$$

and the solution would be

$$y = \int f(x)\,\mathrm{d}x + C,$$

where C is an arbitrary constant. On the other hand, if f is a function of only y, then Equation (5.4) reduces to

$$y' = f(y),$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y),$$

whose variables can be separated as

$$\frac{\mathrm{d}y}{f(y)} = \mathrm{d}x,$$

and, in principle, integrating both sides yields the solution.

Example 5.9. Solve the following initial value problem,

$$\begin{cases} \mathrm{i}\hbar \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = E\phi \\ \\ \quad \phi(0) = 5, \end{cases}$$

where E and $\hbar$ are real constants and i is the imaginary unit.

Solution. As mentioned above, this equation can be written as

$$\frac{\mathrm{d}\phi}{\phi} = \frac{E}{\mathrm{i}\hbar}\mathrm{d}t.$$

Integrating both sides, we get

$$\ln|\phi(t)| = -\frac{\mathrm{i}}{\hbar}E\,t + C,$$

or

$$\begin{aligned}\phi(t) &= \pm\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E\,t+C} \\ &= \pm\mathrm{e}^{C}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E\,t}. \qquad (5.5)\end{aligned}$$

Using the initial condition $\phi(0) = 5$ in the Equation (5.5), yields

$$5 = \pm\mathrm{e}^{C},$$

but since e^{C} is positive, we realize that the solution with negative sign is not acceptable and $\mathrm{e}^{C} = 5$. Therefore, from Equation (5.5)

$$\phi(t) = 5\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E\,t}.$$

◇

Definition 5.7. Any equation in the form of

$$y' + p(x)y = g(x), \qquad (5.6)$$

is called a first order linear ODE.

Theorem 5.1. *The general solution of Equation (5.6) is*

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)g(x)\,\mathrm{d}x + c\right), \qquad (5.7)$$

where

$$\mu(x) = \mathrm{e}^{\int^{x} p(t)\,\mathrm{d}t}, \qquad (5.8)$$

and c is a constant.

Note 5.5. *Formula (5.7) is valid only if the coefficient of y' in Equation (5.6) is equal to 1, if not, one should first divide both sides of this equation by the coefficient of y'.*

Example 5.10. Find the general solution of the equation $xy' + 2y = 4x^2 + 3x$.

Solution. By Note 5.5, we know that, to be able to use Theorem 5.1, first we need to divide both sides of this equation by x. Doing so, we get the following equation

$$y' + \frac{2}{x}y = 4x + 3,$$

so $p(x) = \frac{2}{x}$ and $g(x) = 4x + 3$. By Equation (5.8),

$$\begin{aligned} \mu(x) &= \mathrm{e}^{\int^x \frac{2}{t}\,\mathrm{d}t} \\ &= \mathrm{e}^{2\ln x} \\ &= x^2, \end{aligned}$$

and finally, by Equation (5.7),

$$\begin{aligned} y &= \frac{1}{x^2}\left(\int x^2(4x+3)\,\mathrm{d}x + c\right) \\ &= \frac{1}{x^2}\left(4\int x^3\,\mathrm{d}x + 3\int x^2\,\mathrm{d}x + c\right) \\ &= \frac{1}{x^2}(x^4 + x^3 + c) \\ &= x^2 + x + \frac{c}{x^2}. \end{aligned}$$

◇

Second Order Ordinary Differential Equations

In this section we will study a small portion of the complicated theory of second order differential equations namely, second order linear differential equations.

Definition 5.8. Any equation of the general form

$$P(x)y'' + Q(x)y' + R(x)y = G(x), \tag{5.9}$$

is called a *second order linear* differential equation.

If $G(x) = 0$, then (5.9) is called a *homogeneous second order linear* differential equation.

Example 5.11. Each of the following equations, is a homogeneous second order linear differential equation:

1. $y'' - 2xy' + 2py = 0,$ (Hermite equation of order p)

2. $(1 - x^2)y'' - 2xy' + l(l + 1)y = 0,$ (Legendre equation of order l)

3. $x^2y'' + xy' + (x^2 - \nu^2)y = 0.$ (Bessel equation of order ν)

We will only study the homogeneous second order differential equations. These are the only ones that one would need in this course. The general method for solving of equations is the method of power series. This method works based on the assumption that the solution of the equation has a power series expansion around some point, which is valid in an open interval around that point. We do not address the questions related to the convergence of the power series or the existence of them for a given equation, since in all the equations of this sort that we deal with in this course, the abstract mathematical prerequisites are fulfilled.

Before starting to investigate the most general case, using the power series method, there is an important class of these equations that is very important in quantum mechanics and is very easy to solve. These are homogeneous second order linear differential equations with *constant coefficient* of the general form $ay'' + by' + cy = 0$ where a, b, and c are just constant real numbers and $a \neq 0$.

Definition 5.9. The quadratic equation $ar^2 + br + c = 0$ is called the *characteristic* equation of $ay'' + by' + c = 0$.

Theorem 5.2. *Consider the differential equation*

$$ay'' + by' + cy = 0 \tag{5.10}$$

and let $\Delta = b^2 - 4ac$.

1. *If* $\Delta = 0$, *then the characteristic equation of (5.10) has one real repeated root* r *and the general solution of (5.10) is*

$$y = (c_1 + c_2x)\mathrm{e}^{rx}. \tag{5.11}$$

2. *If* $\Delta \neq 0$, *then the characteristic equation of (5.10) has two roots,* r_1 *and* r_2[§], *and the general solution of (5.10) is*

$$y = c_1\mathrm{e}^{r_1x} + c_2\mathrm{e}^{r_2x}. \tag{5.12}$$

[§]which are either both real or both complex, depending on the sign of Δ.

Note 5.6. *In the case of negative Δ, the characteristic equation $ar^2+br+c=0$ has two roots which are complex conjugate to each other. Assume that $r_1 = \alpha+\mathrm{i}\beta$ and $r_2 = \alpha - \mathrm{i}\beta$. Then one can show that the general solution of the Equation (5.10) can also be written as follows,*

$$y = \mathrm{e}^{\alpha x}(d_1 \cos \beta x + d_2 \sin \beta x), \tag{5.13}$$

for some constants d_1 and d_2.

Example 5.12. Find the general solution of the following differential equations:

1. $y'' + 5y' + 6 = 0$,
2. $y'' - 4y' + 4 = 0$,
3. $y'' - 2y' + 5y = 0$.

Solution.

1. The characteristic equation for this equation is $r^2 + 5r + 6 = 0$ which has two real roots $r_1 = -2$ and $r_2 = -3$. Thus, by Eqaution (5.12), the general solution of this equation is
$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-3x}.$$
2. For this case, the characteristic equation is $r^2 - 4r + 4 = 0$ which has only one repeated root $r = 2$. So, by Equation (5.11), the general solution is
$$y = (c_1 + c_2 x)\mathrm{e}^{2x}.$$
3. In this case, the characteristic equation is $r^2 - 2r + 5 = 0$ which possesses two complex roots $r_1 = 1 + 2\mathrm{i}$ and $r_2 = 1 - 2\mathrm{i}$. So using Equation (5.12), the general solution can be written as
$$y = c_1\mathrm{e}^{(1+2\mathrm{i})x} + c_2\mathrm{e}^{(1-2\mathrm{i})x},$$
or, alternatively, using Equation (5.13), it can be expressed as
$$y = \mathrm{e}^{x}(d_1 \cos 2x + d_2 \sin 2x).$$

◇

Now let us go back to the main problem, namely, the problem of finding the solutions of the most general homogeneous second order linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{5.14}$$

using power series as our main tool. Before that, let us go briefly through the definition and properties of infinite power series.

Definition 5.10. An infinite series

$$\sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \tag{5.15}$$

where a_n's are constants and x is a variable, is called a *power series*[§].

The following theorem specifies the status of convergence of an infinite power series in all possible cases.

Theorem 5.3. *The convergence of the series* $\sum_{n=0}^{+\infty} a_n x^n$ *is described by one of the following three cases:*

1. *There is a positive number R such that the series diverges for x with $|x| > R$ but converges absolutely for x with $|x| < R$*[¶]*.*
2. *The series converges absolutely for every x.*
3. *The series converges at $x = 0$ and diverges elsewhere.*

Note 5.7. *Roughly speaking, Theorem 5.3 asserts that the set of real numbers x for which the series (5.15) converges, constitutes an interval. In the first case, this interval can take on one of the following forms:*

$$(-R, R) \ , \ [-R, R) \ , \ (-R, R] \ , \ [-R, R]. \tag{5.16}$$

In the second case this interval is $(-\infty, +\infty)$ and finally for the last case this interval is the degenerate interval $[0, 0] = \{0\}$.

Definition 5.11. Depending on the convergence status of the power series (5.15), the appropriate interval mentioned in Note 5.7, is called the *interval of convergence* of the series.

[§]To be precise, this is a power series about 0, but this is the only kind of power series that we need.

[¶]The series may or may not converge at either of the endpoints $x = -R$ and $x = R$.

Definition 5.12. The positive real number R mentioned in Theorem 5.3, is called the *radius of convergence* of the series. In case 2, the radius of convergence is said to be *infinite* and in the last case it is said to be *zero*.

Definition 5.13. Let I be the interval of convergence of the series (5.15) and let f be a function such that for any x in I,

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n. \tag{5.17}$$

Then it is said that this series *represents* f on I.

Theorem 5.4. *Let $\sum_{n=0}^{+\infty} a_n x^n$ and $\sum_{n=0}^{+\infty} b_n x^n$ be two infinite power series with radii of convergence R_1 and R_2, respectively, and let f and g be functions represented by these power series on their corresponding intervals of convergence. If $R = \min\{R_1, R_2\}$, then for any x in the interval $(-R, R)$,*

$$f(x) \pm g(x) = \sum_{n=0}^{+\infty} (a_n \pm b_n) x^n. \tag{5.18}$$

Theorem 5.5. *Let $\sum_{n=0}^{+\infty} a_n x^n$ be an infinite power series with radius of convergence R and let f be the function represented by this series on its interval of convergence. Then the function f is continuous and differentiable of all orders on the interval $(-R, R)$. Moreover, one can determine f', f'', f''', ... by term by term differentiation of this series and all the resultant series are convergent on the interval $(-R, R)$.*

Note 5.8. *Practically, Theorem 5.5 asserts that, if for any x in the interval $(-R, R)$,*

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{+\infty} a_n x^n,$$

then for every x in the same interval $(-R, R)$,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{+\infty} n a_n x^{n-1},$$

and

$$f''(x) = 2a_2 + 6a_3 x + \cdots + n(n-1) a_n x^{n-2} = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2},$$

and so on.

Theorem 5.6. *Suppose that* $\sum_{n=0}^{+\infty} a_n x^n$ *represents a function* f *on its interval of convergence. Then for any* n*,*

$$a_n = \frac{f^{(n)}(0)}{n!}\cdot \tag{5.19}$$

Theorem 5.7. *Let* $\sum_{n=0}^{+\infty} a_n x^n$ *and* $\sum_{n=0}^{+\infty} b_n x^n$ *be two infinite power series with radii of convergence* R_1 *and* R_2*, respectively, and let* $R = \min\{R_1, R_2\}$*. Suppose that for every* x *in* $(-R, R)$*,*

$$\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} b_n x^n.$$

Then for all n*,* $a_n = b_n$*.*

Corollary 5.1. *Let* $\sum_{n=0}^{+\infty} a_n x^n$ *be an infinite power series with interval of convergence* $(-R, R)$ *and for any* x *in this interval*

$$\sum_{n=0}^{+\infty} a_n x^n = 0.$$

Then for all n*,* $a_n = 0$*.*

Before leaving this discussion let us mention two properties of summations.

Note 5.9. *The index of a summation notation is a dummy index and changing it does not alter anything, i.e. for example,*

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!} = \sum_{k=0}^{+\infty} \frac{x^k}{k!} = \sum_{i=0}^{+\infty} \frac{x^i}{i!} = \cdots.$$

Note 5.10. *Let* k *and* p *be integers and* f *be an arbitrary function. Then*

$$\sum_{n=k}^{+\infty} f(n) = \sum_{n=k\pm p}^{+\infty} f(n \mp p). \tag{5.20}$$

For example

$$\sum_{n=2}^{+\infty} (n+1)(n+2) a_n x^{n-2} = \sum_{n=0}^{+\infty} (n+3)(n+4) a_{n+2} x^n.$$

For getting familiar with the general aspects of power series method for solving the differential equations, let us solve the third part of Example 5.12 again, but this time using a power series about 0 as a candidate of the solution.

Assume that $y'' + y = 0$ has a solution in the form of a infinite power series[§], $y = \sum_{n=0}^{+\infty} a_n x^n$. Then, as mentioned in Note 5.8, $y'' = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2}$. Therefore, plugging these relations back into the equation we end up with

$$\sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{+\infty} a_n x^n = 0, \tag{5.21}$$

which, using Equation (5.20), can be written as

$$\sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{+\infty} a_n x^n = 0, \tag{5.22}$$

or more compactly,

$$\sum_{n=0}^{+\infty} [(n+1)(n+2)a_{n+2} + a_n] x^n = 0. \tag{5.23}$$

From Corollary 5.1, one concludes that for any non-negative integer n,

$$(n+1)(n+2)a_{n+2} + a_n = 0, \tag{5.24}$$

or

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}\, a_n. \tag{5.25}$$

The relation in Equation (5.25) is kind of a, so-called, *recursive* relation which relates a higher term of a sequence, in this case a_{n+2}, to a lower term/terms, which is a_n in this case, of the same sequence. As I will show, this relation enables us to determine all terms of the sequence $\{a_n\}$ in terms of just the first two terms a_0 and a_n. Substituting 0 and 1 for n in Equation (5.25), yields,

$$a_2 = -\frac{1}{1.2}a_0 = -\frac{1}{2!}a_0 \quad , \quad a_3 = -\frac{1}{2.3}a_1 = -\frac{1}{3!}a_1 \tag{5.26}$$

Assigning 2 and 3 to n in Equation (5.25), bring us to the following relations

$$a_4 = -\frac{1}{3.4}a_2 \quad , \quad a_5 = -\frac{1}{4.5}a_3, \tag{5.27}$$

[§]The validity of this assumption is based on some mathematical theorems, which we skipped.

which after applying relations (5.26) to them we end up with

$$\begin{aligned} a_4 &= \left(-\frac{1}{3.4}\right)\left(-\frac{1}{2!}a_0\right) \quad , \quad & a_5 &= \left(-\frac{1}{4.5}\right)\left(-\frac{1}{3!}a_1\right) \\ &= +\frac{1}{4!}a_0 & &= +\frac{1}{5!}a_1 \end{aligned}$$

Continuing in this manner one can easily convince oneself that for any non-negative integer n

$$a_{2n} = (-1)^n \frac{1}{(2n)!} a_0 \quad , \quad a_{2n+1} = (-1)^n \frac{1}{(2n+1)!} a_1. \tag{5.28}$$

But separating even and odd terms in the solution $y = \sum_{n=0}^{+\infty} a_n x^n$, and making use of Equations (5.28), we get the following

$$\begin{aligned} y &= \sum_{n=0}^{+\infty} a_{2n} x^{2n} + \sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1} \\ &= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n)!} a_0 x^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n+1)!} a_1 x^{2n+1} \\ &= a_0 \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \\ &= a_0 \sin x + a_1 \cos x, \end{aligned} \tag{5.29}$$

where in the last step, I used the Taylor expansions for $\sin x$ and $\cos x$,

$$\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad , \quad \sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

This is in complete agreement with the answer we got in the third part of Example 5.12. Of course, in this case, the power series method is not recommended!

Actually, the last example reveals almost all *general* aspects of the power series method for solving a homogeneous second order linear differential equation. Let us look at a more complicated problem, for which the power series method is not just an alternative but the only analytic method available.

Example 5.13. Consider the Hermite equation

$$y'' - 2xy' + 2py = 0, \tag{5.30}$$

where p is a constant.

1. Find the general solution of Hermite equation.
2. If p is a non-negative integer, what can one say about the general solution?

Solution. To anwer the first part, assume that this equation has a solution in the form of a power series $y = \sum_{n=0}^{+\infty} a_n x^n$. Plugging this into Equation (5.30) and using what is mentioned in Note 5.8, we get

$$\sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{+\infty} n a_n x^{n-1} + 2p \sum_{n=0}^{+\infty} a_n x^n = 0, \tag{5.31}$$

which can also be written as

$$\sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{+\infty} 2n a_n x^n + \sum_{n=0}^{+\infty} 2p a_n x^n = 0, \tag{5.32}$$

which based on Note 5.10, this can be re-written as

$$\sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{+\infty} 2n a_n x^n + \sum_{n=0}^{+\infty} 2p a_n x^n = 0. \tag{5.33}$$

The middle-term sum in Equation (5.33) can start from $n = 0$ instead of $n = 1$ since $2na_n x^n$ is zero for $n = 0$. Hence, collecting the three terms, we have

$$\sum_{n=0}^{+\infty} [(n+1)(n+2)a_{n+2} + 2(p-n)a_n]x^n = 0. \tag{5.34}$$

Therefore, for all non-negative integers n,

$$(n+1)(n+2)a_{n+2} + 2(p-n)a_n = 0,$$

or

$$a_{n+2} = -\frac{2(p-n)}{(n+1)(n+2)} a_n. \tag{5.35}$$

For $n = 0$ and $n = 1$, Equation (5.35) yields,

$$a_2 = -\frac{2p}{1.2} a_0 = -\frac{2p}{2!} a_0 \quad , \quad a_3 = -\frac{2(p-1)}{2.3} a_1 = -\frac{2(p-1)}{3!} a_1. \tag{5.36}$$

Substituting $n = 2$ and $n = 3$ in Equation (5.35), we get

$$a_4 = -\frac{2(p-2)}{3.4} a_2 \qquad , \quad a_5 = -\frac{2(p-3)}{4.5} a_3$$

$$= \left(-\frac{2(p-2)}{3.4}\right)\left(\frac{-2p}{2!}a_0\right) \qquad = \left(-\frac{2(p-3)}{4.5}\right)\left(-\frac{2(p-1)}{3!}a_1\right)$$
$$= +\frac{2^2p(p-2)}{4!}a_0 \qquad = +\frac{2^2(p-1)(p-3)}{5!}a_1.$$

In the same manner one can see,

$$a_6 = -\frac{2^3p(p-2)(p-4)}{6!}a_0,$$

and

$$a_7 = -\frac{2^3(p-1)(p-3)(p-5)}{7!}a_1,$$

and so on. Now the solution $y = \sum_{n=0}^{+\infty} a_n x^n$ would be

$$\begin{aligned}
y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \\
&= a_0 + a_1x - \frac{2p}{2!}a_0x^2 - \frac{2(p-1)}{3!}a_1x^3 + \frac{2^2p(p-2)}{4!}a_0x^4 \\
&\quad + \frac{2^2(p-1)(p-3)}{5!}a_1x^5 - \frac{2^3p(p-2)(p-4)}{6!}a_0x^6 \\
&\quad - \frac{2^3(p-1)(p-3)(p-5)}{7!}a_1 + \cdots \\
&= a_0\left(1 - \frac{2p}{2!}x^2 + \frac{2^2p(p-2)}{4!}x^4 - \frac{2^3p(p-2)(p-4)}{6!}x^6 + \cdots\right) \\
&\quad + a_1\left(x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5 - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 + \cdots\right),
\end{aligned}$$

or

$$\begin{aligned}
y =& a_0\left(1 + \sum_{n=1}^{+\infty}(-1)^n\frac{2^np(p-2)\cdots(p-2n+2)}{(2n)!}x^{2n}\right) \\
&+ a_1\left(x + \sum_{n=1}^{+\infty}(-1)^n\frac{2^n(p-1)(p-3)\cdots(p-2n+1)}{(2n+1)!}x^{2n+1}\right), \qquad (5.37)
\end{aligned}$$

where a_0 and a_1 are arbitrary constants. This is the general solution of Hermite differential equation. As is expected from a second order differential equation, this solution consists of two linearly independent parts, where each of them, in general, is an infinite power series. Of course, there is an exception and that is the case where p is a non-negative integer. If p is a non-negative even integer, it is clear that the first series in Equation (5.37) terminates eventually and it would be an even polynomial of degree p. But, of course, in this case the second term would be an infinite power series. On the other hand, if p is a positive odd integer, the second term would terminate and it would be an odd polynomial of degree p but in this case the first term would be an infinite power series. So to address the second part of this example, one can say that if p is a non-integer or a negative integer, there are two independent solutions to the Hermite Equation (5.30), both of them having infinite number of terms. If p is a non-negative integer, then this equation has two independent solutions such that one of them is an even or an odd polynomial, depending on p being odd or even, of degree p and the other one is a power series with infinite number of terms[§]. $\diamond$

Sometimes physical considerations force us to be interested, not in all but in some specific class of solutions of an ODE. In quantum mechanics, for example, this physical constraint could be that the wave function[¶] ψ should go to zero far from the origin. Very often, taking advantage of these physical conditions, makes it easier to solve the corresponding mathematical problem. To illuminate this point, let us go over an example thoroughly.

Consider the differential equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (3 - x^2)\psi = 0, \tag{5.38}$$

and suppose that based on some physical assumptions, we are only interested in those solutions ψ of this equation such that

$$\lim_{x\to\pm\infty} \psi(x) = 0. \tag{5.39}$$

As we will see, if we don't care about this constraint from the beginning and try to solve this equation by the power series method first and try to impose condition (5.39) on the solution afterwards, we will run into mathematical subtleties. To see this very clear, let us assume that Equation (5.38) has a solution of the form $\psi = \sum_{n=0}^{+\infty} a_n x^n$. Hence, $\psi'' = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2}$. Substituting these relations in Equation (5.38), one gets

$$\sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{+\infty} 3a_n x^n - \sum_{n=0}^{+\infty} a_n x^{n+2} = 0,$$

[§]Hermite equation plays a fundamental role when dealing with the harmonic oscillator problem in quantum mechanics.

[¶]If you don't know what a wave function is, don't worry at this stage. Consider it just as a mathematical function.

or,

$$\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{+\infty} 3a_n x^n - \sum_{n=2}^{+\infty} a_{n-2}x^n = 0.$$

Writing down the first two terms in the first and second sum of the above equation explicitly, we get

$$\left[2a_2 + 6a_3 x + \sum_{n=2}^{+\infty}(n+1)(n+2)a_{n+2}x^n\right] + \left[3a_0 + 3a_1 x + \sum_{n=2}^{+\infty} 3a_n x^n\right] - \sum_{n=2}^{+\infty} a_{n-2}x^n = 0,$$

or

$$(3a_0 + 2a_2) + 3(a_1 + 2a_2)x + \sum_{n=2}^{+\infty}[(n+1)(n+2)a_{n+2} + 3a_n - a_{n-2}]x^n = 0,$$

which yields the following equations,

$$3a_0 + 2a_2 = 0, \tag{5.40}$$

and

$$a_1 + 2a_3 = 0, \tag{5.41}$$

and for all integers $n \geq 2$,

$$(n+1)(n+2)a_{n+2} + 3a_n - a_{n-2} = 0. \tag{5.42}$$

Although, in principle, Equations (5.40), (5.41), and (5.42) enable one to determine all the coefficients a_n and, consequently, the solution $\psi = \sum_{n=0}^{+\infty} a_n x^n$, but finding a closed form for the solution using Equation (5.42) which relates together *three* different terms of the sequence a_n would be difficult. Therefore, it is not clear how one can apply the condition (5.39) on a solution that its closed form is not known.

Now let us look at the asymptotic behavior of Equation (5.38) as $x \to \pm\infty$, which is, of course, a standard technique in this kind of problems. In this case 3 would be negligible compared with x^2 so this equation reduces, approximately, to the equation in Example 5.5, which based on that example, has the solutions

$$\psi(x) = \mathrm{e}^{\pm\frac{x^2}{2}}. \tag{5.43}$$

But now,

$$\psi' = \pm x \mathrm{e}^{\pm\frac{x^2}{2}},$$

and

$$\psi'' = x^2\mathrm{e}^{\pm\frac{x^2}{2}} \pm \mathrm{e}^{\pm\frac{x^2}{2}}.$$

Since for large x the second term of ψ'' can be neglected compared with the first one, it appears that $\psi = \mathrm{e}^{\frac{x^2}{2}}$ and $\psi = \mathrm{e}^{-\frac{x^2}{2}}$ are indeed "approximate solutions" of Equation (5.38). The first of these should be discarded since it does not fulfill the constraint Equation (5.39). It is therefore reasonable to suppose that the exact solution of Equation (5.38) has the form

$$\psi(x) = y(x)\mathrm{e}^{-\frac{x^2}{2}}. \tag{5.44}$$

Therefore, the condition (5.39) reduces to the following condition

$$\lim_{x\to\pm\infty} y(x)\mathrm{e}^{-\frac{x^2}{2}} = 0. \tag{5.45}$$

Taking off the exponential part, we hope that the function $y(x)$ has a simpler functional form than $\psi(x)$ itself. Consequently,

$$\begin{aligned}\psi' &= y'\mathrm{e}^{-\frac{x^2}{2}} - xy\mathrm{e}^{-\frac{x^2}{2}}\\ &= (y' - xy)\mathrm{e}^{-\frac{x^2}{2}},\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}\psi'' &= (y'' - y - xy')\mathrm{e}^{-\frac{x^2}{2}} - x(y' - xy)\mathrm{e}^{-\frac{x^2}{2}}\\ &= (y'' - y - 2xy' + x^2y)\mathrm{e}^{-\frac{x^2}{2}}.\end{aligned} \tag{5.47}$$

Plugging Equations (5.44) and (5.47) back into the Equation (5.38), yields

$$(y'' - y - 2xy' + x^2y)\mathrm{e}^{-\frac{x^2}{2}} + (3 - x^2)ye^{-\frac{x^2}{2}} = 0, \tag{5.48}$$

or after dividing by $\mathrm{e}^{-\frac{x^2}{2}}$ and very little algebraic manipulation yields,

$$y'' - 2xy' + 2y = 0, \tag{5.49}$$

which is the Hermite's Equation (5.30) corresponding to the odd integer $p = 1$. Thus, using Equation (5.37), there exist two solutions,

$$y_1(x) = x, \tag{5.50}$$

and

$$y_2(x) = 1 - \sum_{n=1}^{+\infty} 2^n \, \frac{1.3.5 \cdots (2n-3)}{(2n)!} \, x^{2n} \tag{5.51}$$

But

$$\begin{aligned} 2^n \, \frac{1.3.5 \cdots (2n-3)}{(2n)!} &= 2^n \, \frac{1.3.5 \cdots (2n-3)}{1.2.3.4.5 \cdots .(2n-3)(2n-2)(2n-1)(2n)} \\ &= 2^n \, \frac{1}{[2.4.6 \cdots (2n-2)(2n)](2n-1)} \\ &= 2^n \, \frac{1}{2^n (1.2.3 \cdots n)(2n-1)} \\ &= \frac{1}{(2n-1)n!}, \end{aligned}$$

so

$$y_2(x) = 1 - \sum_{n=1}^{+\infty} \frac{1}{(2n-1)n!} \, x^{2n}. \tag{5.52}$$

Apparently, the solution (5.50) fulfills the condition (5.45) so it is acceptable and therefore, using Equation (5.44), an acceptable solution of Equation (5.38) is

$$\psi(x) = x \mathrm{e}^{-\frac{x^2}{2}}. \tag{5.53}$$

On the other hand, we show that Equation (5.52) does not satisfy the condition (5.45). Therefore, (5.53) is the only solution of Equation (5.38) which fulfills the constraint (5.39). Our strategy is to show that

$$\lim_{x \to \pm\infty} \frac{y_2(x)}{\mathrm{e}^{\frac{x^2}{2}}} \tag{5.54}$$

cannot be zero. Consider the Taylor expansion of $\mathrm{e}^{\frac{x^2}{2}}$,

$$\begin{aligned}\mathrm{e}^{\frac{x^2}{2}} &= \sum_{n=0}^{+\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} \\ &= 1 + \sum_{n=1}^{+\infty} \frac{1}{2^n n!}\, x^{2n}. \end{aligned} \tag{5.55}$$

But for all positive integers n,

$$\frac{1}{(2n-1)n!} > \frac{1}{2^n n!},$$

so (5.54) cannot be zero.

5.2 Partial Differential Equations

In this section I define what a partial differential equation is and by investigating a typical example thoroughly, I try to illuminate the general aspects of the method of separation of variables for solving this kind of equations.

Definition 5.14. An equation consisting of an unknown function with more than one variable and its partial derivatives is called a *partial differential equation* or shortly *PDE*.

Example 5.14. The following equations are partial differential equations:

1. $\dfrac{\partial^2 \phi}{\partial t^2} - 4\,\dfrac{\partial^2 \phi}{\partial x^2} = 0,$ (Wave Equation)

2. $\mathrm{i}\hbar\,\dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\Psi + V(x,y,z,t)\Psi,$ (Schrödinger Equation)

where in the first one, the unknown function is ϕ which is a function of x and t and in the second one the function we are looking for is Ψ and it is a function of x, y, z, and t. i is the complex unit, $\hbar$ is a positive constant which is called *Planck's* constant and ∇^2 refers to the Laplacian operator in Cartesian coordinates, that is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The first equation is of second order in x and t and the second equation is of order one in t and it is of second order in other variables.

Example 5.15. Show that the function

$$u(x,t) = \sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{x^2}{4\alpha^2 t}}$$

is a solution of the following equation

$$\alpha^2 u_{xx} = u_t,$$

where $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ and $u_t = \frac{\partial u}{\partial t}$.

Solution.

$$\begin{aligned} u_x &= -\frac{2x}{4\alpha^2 t}\sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{x^2}{4\alpha^2 t}} \\ &= -\frac{1}{2\alpha^2 t}\, xu, \end{aligned}$$

so

$$\begin{aligned} u_{xx} &= -\frac{1}{2\alpha^2 t}(u + xu_x) \\ &= -\frac{1}{2\alpha^2 t}\left(u - \frac{1}{2\alpha^2 t}x^2 u\right), \end{aligned}$$

so

$$\alpha^2 u_{xx} = \left(\frac{x^2}{4\alpha^2 t^2} - \frac{1}{2t}\right)u. \tag{5.56}$$

On the other hand,

$$u_t = -\frac{1}{2t}\sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{x^2}{4\alpha^2 t}} + \sqrt{\frac{\pi}{t}}\frac{x^2}{4\alpha^2 t^2}\mathrm{e}^{-\frac{x^2}{4\alpha^2 t}},$$

or

$$u_t = \left(\frac{x^2}{4\alpha^2 t^2} - \frac{1}{2t}\right)u. \tag{5.57}$$

Now it suffices to compare both sides of Equations (5.56) and (5.57). $\diamond$

The only method that is used to solve a PDE in this course is the method of *separation of variables* which now I am going to explain its general aspects by looking at a typical example, namely, the vibrating string in full detail.

Example 5.16. A string of length L, with ends fixed at $x = 0$ and $x = L$, is pulled upward at the middle so it reaches height h. What is the subsequent motion of the string if it is released from the rest?

Solution. Before starting to solve this problem, let us find the function which represents the shape of the string at time $t = 0$. Note that if this string is pulled upward at the middle to height h, with two ends being fixed at $x = 0$ and $x = L$, it forms a triangle with vertices at $(0, 0)$, $(L, 0)$, and $(L/2, h)$. So the initial shape of the string can be described by the function

$$f(x) = \begin{cases} \dfrac{2h}{L}x, & 0 \leq x \leq \dfrac{L}{2} \\ \dfrac{2h}{L}(L - x), & \dfrac{L}{2} \leq x \leq L. \end{cases} \tag{5.58}$$

By Newton's second law, one can see that the equation governing the motion of such string for small vibrations is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\,\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{5.59}$$

where a is a constant depending on the the density and the tension on the string and $u(x, t)$ is the vertical displacement, at time t, of the point on the string that is at distance x away from the first end point $x = 0$. But the problem imposes some constraints on the solution $u(x, t)$. The end points are fixed. This implies

$$u(0, t) = u(L, t) = 0, \tag{5.60}$$

for *all* time t. As we saw, the initial shape of the string is described by the function $f(x)$, that is

$$u(x, 0) = f(x), \tag{5.61}$$

for all x and since this string is released from the rest, the initial (vertical) velocity of each particle of the string should be zero, that is

$$\frac{\partial u}{\partial t}(x, 0) = 0, \tag{5.62}$$

for any x. So our goal is to find a solution of Equation (5.59) which fulfills the conditions (5.60), (5.61), and (5.62). But before that, there is one aspect of this equation that is essential for what follows and worth mentioning.

This equation is a linear partial differential equation. One important consequence of that which can be easily demonstrated by direct substitution is that if $u_1(x, t)$ and $u_2(x, t)$ are solutions of this equation satisfying the boundary conditions (5.60) and the initial condition (5.62), and c_1 and c_2 are two arbitrary constants, the linear combination $u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)$ is also a solution of this equation satisfying the same boundary and initial conditions.

Now we want to use the method of separation of variables to find the solution of this equation[§]. That is, we assume that the solution $u(x,t)$ can be written as a product of a function $X(x)$ of x alone and a function $T(t)$ of t alone[¶],

$$u(x,t) = X(x)T(t). \tag{5.63}$$

Before going any further, let us take care of the boundary conditions (5.60) and the initial condition (5.62) first[‡]. Using (5.60) in (5.63) we see that for *all* instants of time t,

$$X(0)T(t) = 0,$$

and

$$X(L)T(t) = 0,$$

or consequently,

$$X(0) = X(L) = 0, \tag{5.64}$$

and using (5.62) in (5.63) one finds out, for *all* x, that

$$X(x)\frac{\mathrm{d}T}{\mathrm{d}t}(0) = 0$$

or

$$T'(0) = 0, \tag{5.65}$$

where the prime on T represents its derivative with respect to t. Substituting (5.63) for $u(x,t)$ in Equation (5.59), one gets

$$T\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = \frac{1}{a^2}\,X\frac{\mathrm{d}^2T}{\mathrm{d}t^2},$$

and after dividing both sides of this equation by XT,

$$\frac{1}{X}\,\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = \frac{1}{a^2}\,\frac{1}{T}\,\frac{\mathrm{d}^2T}{\mathrm{d}t^2}. \tag{5.66}$$

[§]Of course, $u(x,t) = 0$ is a solution of Equation (5.59), but apparently this solution does not satisfy the initial condition $u(x,0) = f(x)$.

[¶]The validity of this assumption become clear at the end, when we have actually found the solution!

[‡]Taking care of the initial condition (5.61) is mathematically more subtle and we consider it at final steps of the solution.

The left hand side of Equation (5.66) is a function only of x and its right hand side is a function only of t. So this equality cannot hold unless both sides are equal to a constant, since otherwise by keeping, say, t fixed and making a change in x, in general the left hand side varies but the right hand side does not which is absurd. Let us call this constant, μ[§]. Therefore a *single* partial differential equation (5.59), reduces into *two* ordinary differential equations

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = \mu,$$

and

$$\frac{1}{a^2}\frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \mu,$$

or

$$X'' - \mu X = 0, \tag{5.67}$$

and

$$T'' - \mu a^2 T = 0, \tag{5.68}$$

where prime on X stands for differentiation with respect to x and prime on T represents differentiation with respect to t.

Now we want to know which values of μ give rise to an acceptable separable solution $u(x,t) = X(x)T(t)$ for Equation (5.59), that is, a separable solution which satisfies the Equations (5.64) and (5.65)[¶]. We show that μ should be a negative real number and, in fact, all negative values of μ correspond to an acceptable *separable* solution. The reason is as follows. For $\mu = 0$, the Equation (5.65) reduces to

$$X''(x) = 0,$$

with the general solution

$$X(x) = c_1 x + c_2. \tag{5.69}$$

But applying the Equations (5.64) on (5.69) yields $c_1 = c_2 = 0$ and therefore $X(x) = 0$, or as a result, $u(x,t) = 0$ which is not acceptable. Thus, μ cannot be zero.

[§] At this stage, we do not know anything about μ, except that it is a real number.

[¶] For the time being, do not concern about the the initial condition (5.62). As it will be clear in a while, there exist no *separable* solution of Equation (5.59) which holds all boundary and initial conditions simultaneously. We need a trick, Fourier trick, to overcome this problem.

Now let us suppose that μ is a positive number. Then the characteristic equation[§] of the Equation (5.67), which is

$$r^2 - \mu = 0, \tag{5.70}$$

has two real roots $-\sqrt{\mu}$ and $+\sqrt{\mu}$, and according to Theorem (5.2), its general solution is

$$X(x) = c_1 \mathrm{e}^{+\sqrt{\mu}x} + c_2 \mathrm{e}^{-\sqrt{\mu}x}. \tag{5.71}$$

But for this solution, Equations (5.64) imply

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 \mathrm{e}^{+\sqrt{\mu}L} + c_2 \mathrm{e}^{-\sqrt{\mu}L} = 0 \end{cases}. \tag{5.72}$$

Solving this system of equations for c_1 and c_2, also yields $c_1 = c_2 = 0$ which is again not acceptable for the same reason as before[¶].

The only possibility left is to consider negative values for μ. In this case, the characteristic Equation (5.70) has $\mathrm{i}\sqrt{-\mu}$ and $-\mathrm{i}\sqrt{-\mu}$ as its roots, so according to Note 5.7, its general solution can be written as

$$X(x) = c_1 \cos(\sqrt{-\mu}x) + c_2 \sin(\sqrt{-\mu}x). \tag{5.73}$$

Now we apply the boundary conditions (5.64) on this solution.

$$X(0) = 0 \to c_1 \cos(\sqrt{-\mu}0) + c_2 \sin(\sqrt{-\mu}0) = 0 \to c_1 \cos 0 + c_2 \sin 0 = 0 \to c_1 = 0.$$

Therefore,

$$X(x) = c_2 \sin(\sqrt{-\mu}x), \tag{5.74}$$

and now

$$X(L) = 0 \to c_2 \sin(\sqrt{-\mu}L) = 0 \to c_2 = 0 \text{ or } \sin(\sqrt{-\mu}L) = 0.$$

But c_2 cannot be zero, since otherwise from (5.74) one concludes $X(x) = 0$, which is not acceptable as mentioned before. Thus,

$$\sin(\sqrt{-\mu}L) = 0,$$

[§] Refer to Definition 5.9.

[¶] In these two cases, which an acceptable solution for $X(x)$ does not exist, we do not bother considering the other Equation (5.68), since, anyway, the acceptable solution does not exist even if $T(t)$ does.

or

$$\sqrt{-\mu}L = n\pi,$$

where n is an integer. Of course, since $\sqrt{-\mu}L$ is a positive number, n should be a positive integer. Hence, the only values of μ which may be[§] acceptable are μ_n's such that

$$-\mu_n^2 L^2 = n^2\pi^2,$$

or

$$\mu_n = -\frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots. \tag{5.75}$$

Plugging these values for μ in Equation (5.74), one gets

$$X_n(x) = c_2 \sin\left(\frac{n\pi}{L}x\right), \tag{5.76}$$

as the acceptable solution of Equation (5.67) corresponding to μ_n. Equipped with these values for μ we are ready to tackle the Equation (5.68). Let T_n be the solution of Equation (5.68) associated with n, thus

$$T_n'' - \left(-\frac{n^2\pi^2}{L^2}\right)a^2 T_n = 0,$$

or

$$T_n'' + \frac{n^2\pi^2 a^2}{L^2}T_n = 0. \tag{5.77}$$

The characteristic equation of this equation is

$$r^2 + \frac{n^2\pi^2 a^2}{L^2} = 0,$$

which possesses two solutions

$$r = +\mathrm{i}\frac{n\pi a}{L} \;\;,\;\; r = -\mathrm{i}\frac{n\pi a}{L}.$$

Therefore, the general solution of this equation would be

$$T_n(t) = d_1 \cos\left(\frac{n\pi a}{L}t\right) + d_2 \sin\left(\frac{n\pi a}{L}t\right). \tag{5.78}$$

[§] Actually, it turns out that all μ_n's are acceptable, but at this stage it is not clear since we have two other conditions that should be satisfied, and we have not taken them into account yet. But one thing is for sure, μ cannot be any number except the ones given by Equation (5.75).

Now it is the time to take care of initial condition (5.65). First we need to determine $T_n'(t)$. That is

$$T_n'(t) = -d_1 \frac{n\pi a}{L} \sin\left(\frac{n\pi a}{L} t\right) + d_2 \frac{n\pi a}{L} \cos\left(\frac{n\pi a}{L} t\right),$$

and

$$T_n'(0) = d_2 \frac{n\pi a}{L},$$

so $T_n'(0) = 0$ yields $d_2 = 0$ and the solution (5.78) reduces to

$$T_n(t) = d_1 \cos\left(\frac{n\pi a}{L} t\right). \tag{5.79}$$

Thus we found infinite number of separable solutions

$$\begin{aligned} u_n(x,t) &= X_n(x) T_n(t) \\ &= c_2 d_1 \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right), \end{aligned} \tag{5.80}$$

which fulfill boundary and initial conditions (5.60) and (5.62), but for all of these solutions we get

$$u_n(x,0) = c_2 d_1 \sin\left(\frac{n\pi}{L} x\right),$$

and it is clear that it is not possible to choose constants c_2 and d_1 in such a way that

$$c_2 d_1 \sin\left(\frac{n\pi}{L} x\right) = f(x),$$

so that the last condition is also satisfied. It seems that we came to a dead end. But wait, we found infinite number of separable solutions, one for each value of n, and, most importantly, the given equation is a linear PDE. Hence, according to the second paragraph on page 50, a linear combination

$$u(x,t) = \sum_{n=1}^{+\infty} a_n u_n(x,t), \tag{5.81}$$

or

$$u(x,t) = \sum_{n=1}^{+\infty} a_n c_2 d_1 \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right) \tag{5.82}$$

of these solutions is also a solution which automatically fulfills boundary and initial conditions (5.60) and (5.62). Equation (5.82) can be written as

$$u(x,t) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right), \tag{5.83}$$

where $b_n = a_n c_2 d_1$ are constants which should be determined. So Equation (5.83), for any choices of coefficients b_n, is a solution of Equation (5.59) which satisfies boundary and initial conditions (5.60) and (5.62). Now we want to determine the constants b_n such that the last condition, i.e., condition (5.62) is also fulfilled. Applying this condition to Equation (5.83), one gets

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{L}x\right). \tag{5.84}$$

This also seems hopeless. Since on the right hand side we have a function which is odd, but on the other side we have $f(x)$ which is not. To overcome this problem, we define an odd periodic function $F(x)$ of period $2L$ such that $F(x)$ coincides with $f(x)$ in the interval $[0, L]$. This is called an odd *extension* [§] of $f(x)$. Now instead of Equation (5.84), consider[¶]

$$F(x) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \tag{5.85}$$

or

$$F(x) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{2n\pi}{2L}x\right). \tag{5.86}$$

Comparing this to series (3.15), it becomes clear that the left hand side of Equation (5.86) is actually the Fourier expansion of the odd periodic function $F(x)$. So using the Equation (3.16),

$$b_n = \frac{4}{2L}\int_0^{2L/2} F(x)\sin\left(\frac{2n\pi}{2L}x\right)\,\mathrm{d}x, \tag{5.87}$$

[§] To be explicit

$$F(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x \le 0, \end{cases}$$

and it repeats itself out side the interval $[-L, L]$. But we do not need such complexities.

[¶] I will give a comment to justify the validity of using $F(x)$ instead of $f(x)$. So don't worry, just carry on.

or

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x, \end{aligned} \tag{5.88}$$

since $F(x) = f(x)$ in the interval $[0, L]$. So

$$\begin{aligned} b_n &= \frac{2}{L}\left[\int_0^{L/2} \frac{2h}{L}x\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x + \int_{L/2}^{L} \frac{2h}{L}(L-x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x\right] \\ &= \frac{8h}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right) \end{aligned}$$

where in both terms integration by parts is used and of course $\sin(n\pi) = 0$.

Now the comment which I promised in the last footnote. We wanted to find b_n's such that Equation (5.84) is established, but we saw that it was not possible and instead we found them in such a way that Equation (5.85) is fulfilled. But note that according to Theorem 3.1 Equation (5.85) is valid where $F(x)$ is continuous, which in this case is the whole real line. Hence, this equation is also valid on the interval $[0, L]$ on which $F(x) = f(x)$, justifying using the Equation (5.85) instead of the Equation (5.84).

Therefore, plugging the values found for b_n's into Equation (5.83), we ultimately find the desired solution for this problem:

$$u(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{+\infty}\frac{1}{n^2}\sin\left(\frac{\pi n}{2}\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right). \tag{5.89}$$

I hope that you are not shocked by seeing this infinite series as the solution of this problem. Of course, we had this kind of solutions even back in ODE's.

Before ending this section, I just want to mention that if you naively multiply both sides of Equation (5.84) by $\sin\left(\frac{m\pi}{L}x\right)$ where m is a positive integer and integrate both sides from 0 to L, and use

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \frac{2}{L}\delta_{mn},$$

where δ_{mn} is the *Kronecker* delta defined as

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n, \end{cases}$$

you will get, for all positive integers m,

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \sum_{n=1}^{+\infty}\frac{2}{L}b_n\delta_{mn},$$

or

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x$$

or by switching back to n

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x.$$

This is the same equation for b_n as in the Equation (5.88). But note that, for the reason previously mentioned, Equation (5.84) cannot occur at all, although the results are exactly the same. Anyway, if you are not that kind of person who cares about mathematical precision, feel relaxed, you can always use this shortcut. $\diamond$